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Portfolio analysis with Sharpe ratios resampled with bootstrapping

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ABSTRACT

In this paper, a portfolio analysis is carried out using the Sharpe ratio to identify the optimal market portfolio. The measure of investment performance with a Sharpe ratio is compared to results obtained with bootstrapped resamples of the Sharpe ratio. The results indicate that the choice of the market portfolio is highly affected by the uncertainty regarding the estimation of the expected returns and the variance-covariance matrix between the returns, that is, the estimation risk associated with these parameters.

KEYWORDS

Portfolio analysis; Bootstrapping

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1. Introduction

The Sharpe ratio is a measure of risk-adjusted return that is used to evaluate the performance of an investment relative to its risk. The Sharpe ratio is calculated as the excess return of an investment divided by the standard deviation of returns. A higher Sharpe ratio indicates that an investment has provided higher returns for the amount of risk taken, compared to other investments with lower Sharpe ratios.

Despite the widespread use of the Sharpe ratio, it is not a perfect measure of investment performance, since only considers the point estimates of average return and volatility of an investment, and hence neglects the uncertainty in the estimation of these. Resampling with bootstrap methods can be used to account for the uncertainty in the estimation of the parameters of the Sharpe ratio. By incorporating the level of uncertainty associated with the estimation of the parameters of the Sharpe ratio through resampling, investors and portfolio managers make informed decisions about how to allocate investment capital in a way that balances risk and reward, considering the uncertainty in the estimation of the mean and variance parameters of the Sharpe ratio itself.

Previous studies have suggested improvements to the Sharpe ratio (Morey and Vinod, 2001), by including hypothesis testing (Ledoit and Wolf, 2008; Auer and Schuhmacher, 2013) or resampling (Boynton and Chen, 2018). Skrepnek and Sahai (2011) suggested the use of bootstrap resampling to deal with the uncertainty in the estimation of the Sharpe ratio, but the results of Skrepnek and Sahai (2011) are based on simulation results and do not consider an application to real data. The innovation of our study is the use of bootstrap resampling to measure in practice and with real data the effects on decision making of considering the traditional Sharpe ratio against a resampled Sharpe ratio. In this paper, portfolio analysis techniques are applied to the monthly closing prices of the shares of Thermal Energy International Inc., Crew Energy Inc, KV Pharmaceutical Co. and N-Viro International Corp., from November 2004 to May 2010 (67 observations). The composition of the market portfolio that offers the most efficient investment in terms of return-risk is identified with the traditional Sharpe ratio, and the results are compared with the results of bootstrapping resamples of the Sharpe ratio. Part 2 of the paper describes the methodology, Part 3 contains the results, and Part 4 concludes.

2. Portfolio Analysis with Resampled Sharpe Ratios

Portfolio selection theory is based on the mean-variance analysis considers only the first two moments of the distribution of asset returns. Define $\mu = \left[E[r_1], E[r_2], \ldots, E[r_i]\right]^T$ as the vector of expected returns and Σ as the covariance matrix of returns with $\sigma_i^2 = Var[r_i]$ and $\sigma_{kl} = Cov[r_k, r_l]$ of a portfolio of i assets with weights $\mathrm{w} = [w_i, w_2, \ldots, w_i]^T$, which represent the fraction of capital investment placed in asset i. By definition, $\mathrm{w}^T 1 = 1$. The expected return and variance of the portfolio are,

$$E\left[r_p\right] = w^T \mu, Var\left[r_p\right]$$

If assets free of risk are not included and assuming unlimited short selling, the minimum variance portfolio solves the minimization problem,

$$\min_{w \in \mathbb{R}} w^T \Sigma w \quad \text{s.t. } \mu_{p,i} = w^T \mu, w^T 1 = 1$$

for an expected return of the portfolio μ_p.

The Sharpe ratio (S) can be used to find the optimal market portfolio among the portfolios that form the efficient frontier:

$$S_i = \frac{\mathrm{w}_i^T \mu - rf}{\sqrt{\mathrm{w}_i^T \Sigma \mathrm{w}_i}}$$

The optimal investment decision is the one that chooses the portfolio with the highest Sharpe ratio, with the $w_i^T \mu$ return, $w_i^T \Sigma w_i$ the variance and the w_i weights of the i portfolio, given an rf risk-free rate. However, the expected returns and the standard deviation of the portfolio are not observable and have to be estimated. Hence, the Sharpe ratio is strictly a random variable that has a point estimator,

$$\hat{S}_i = \frac{\mathrm{w}_i^T \hat{\mu} - rf}{\sqrt{\mathrm{w}_i^T \hat{\Sigma} \mathrm{w}_i}}$$

That is, since $\hat{\mu}$ and $\hat{\Sigma}$ are estimators, $\widehat{S_\iota}$ is a random variable with a probability distribution function. There are analytical results for the moments of this distribution (Bao and Ullah, 2005), but instead of assuming a parametric distribution for $\hat{S}_i$, bootstrapping resampling approximates the sampling distribution of $\hat{S}_i$ by investigating the variation of $\hat{S}_i$ in many pseudo-samples of size n.

If for each i-asset $\tilde{S}_{1,i}, \tilde{S}_{2,i}, \dots, \tilde{S}_{B,i}$ meta-estimators are calculated of the Sharpe ratio with $b = 1,2,\dots,B$ bootstrap resamples of size n, an estimator of the standard deviation of $\hat{S}_i$ will be,

$$\hat{\sigma}_{\hat{S},i} = \sqrt{(B-1)^{-1} \sum_{b=1}^{B} \left(\tilde{S}_{b,i} - \bar{\tilde{S}}_i\right)^2}$$

Note that the $\hat{S}_i$ estimate of S_i can only be obtained on one occasion with a sample of size n, so the bootstrap meta-estimator $\tilde{S}_{B,i}$ provides information about the variation of $\hat{S}_i$ (not S_i), and $\hat{\sigma}_{\hat{S},i}$ can be interpreted as the risk estimation of the Sharpe ratio. If two portfolios have very close Sharpe ratios, an investor will choose the portfolio with lower estimation risk in the Sharpe ratio (lower $\hat{\sigma}_{\hat{S},i}$). Vinod and Morey (2001) also suggested constructing a double Sharpe ratio to evaluate a portfolio. This double Sharpe ratio is the quotient between the Sharpe ratio $\hat{S}_i$ and its estimation risk $\hat{\sigma}_{\hat{S},i}, \hat{S}_i / \hat{\sigma}_{\hat{S},i}$.

Finally, percentile bootstrapping provides $1-\alpha$ interval of non-parametric confidence for S_i from the empirical distribution that results from bootstrap samples $(\tilde{S}_{1,i} - \hat{S}_i, \dots, \tilde{S}_{B,i} - \hat{S}_i)$, choosing $\alpha/2$ and $1-\alpha/2$ quantiles:

$$\left[\tilde{S}(1-\alpha/2) - \hat{S}_i, \tilde{S}(\alpha/2) - \hat{S}_i\right]$$

Hence:

$$P\left(\tilde{S}\left(1-\frac{\alpha}{2}\right) - \hat{S}_i < S_i - \hat{S}_i < \tilde{S}\left(\frac{\alpha}{2}\right) - \hat{S}_i\right)$$

$= P\left(\tilde{S}(1-\alpha/2) < S_i - \hat{S}_i < \tilde{S}(\alpha/2)\right) \approx 1-\alpha$. See Kvam and Vidakovic (2007).

3. Results

Figure 1 shows (i) a fall in the listing prices of shares of Thermal Energy International Inc. and KV Pharmaceutical Co. in the last observations of the sample, and (ii) that the yields of Crew Energy Inc. and N-Viro International Corp. are more volatile than Thermal Energy International Inc. and KV Pharmaceutical Co. With the yields of the shares of these companies, the estimated expected returns vector was calculated to be equal to $\hat{\mu} = [0.0020, 0.0089, -0.0383, 0.0036]$, the sample variance-covariance matrix in turn is,

$$\Sigma = \begin{pmatrix} 0.0672 & 0.0079 & 0.0154 & 0.0090 \\ 0.0079 & 0.0284 & 0.0152 & -0.0015 \\ 0.0154 & 0.0152 & 0.0982 & 0.0162 \\ 0.0090 & -0.0015 & 0.0162 & 0.0407 \end{pmatrix}$$

In order to compare the results with a bootstrapped sample, thousand portfolios with a random composition were simulated (Figure 2). Based on this simulation, an efficient-frontier was calculated with the portfolios that produce the optimal allocation in terms of return-risk. The optimal portfolio of lower risk has a return of 7.40%, and a risk of 42.58% (Table 1).

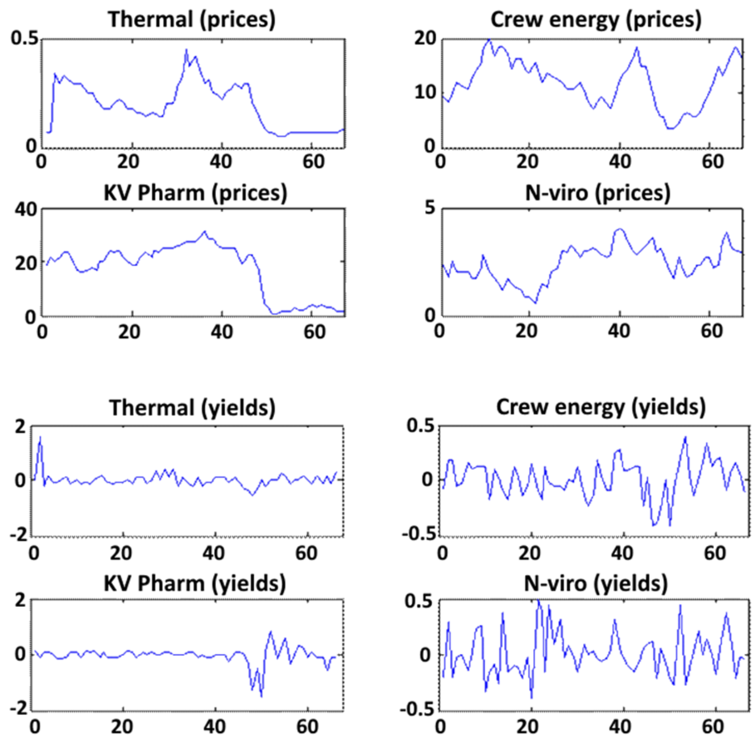

Figure 1. Prices and yields of the assets analyzed: historical information of prices and yields of Thermal, Crew Energy.

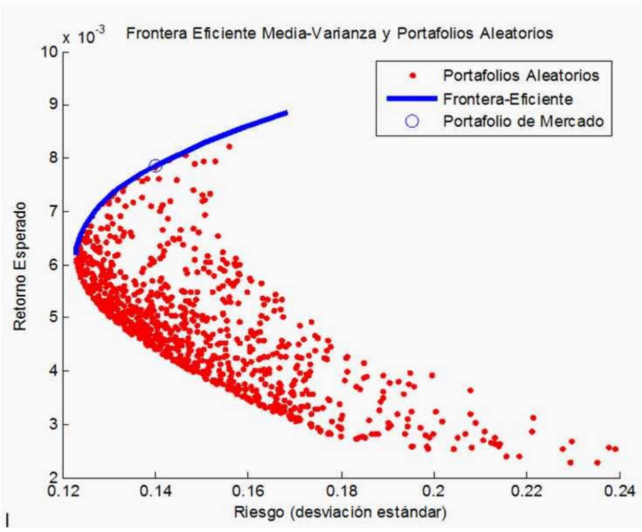

Figure 2. Efficient frontier, random portfolios, and market portfolios.

Assuming a risk-free ratio $rf = 2\%$ (annual), the portfolio with the highest Sharpe ratio is the portfolio number 7, ($\hat{S}_7 = 0.0443$, see Table 1). The annualized performance value of this portfolio is 9.55%, and its risk is 49.26%. The composition that corresponds to this portfolio is $w_7 = [-,0.828,0,0.172]^T$, 0% for Thermal Energy International Inc., 0% for KV Pharmaceutical Co., 82.8% to Crew Energy Inc. and 17.2% to N-Viro International Corp. The results of a thousand resamples with bootstrapping of the Sharpe ratio for this portfolio show that there is considerable uncertainty regarding the value of the punctual estimate of $\hat{S}_7 = 0.0443$, since $\hat{\sigma}_{\hat{S},7} = 0.1128$ and the 95 percent confidence interval includes zero, so it cannot be statistically rejected that the Sharpe ratio of this portfolio is zero (see figure 1 and table 1). Considering the measures of resampling, portfolio 9 has the lowest standard deviation of the Sharpe ratio (lowest estimation risk), equal to $\hat{\sigma}_{\hat{S},9} = 0.0986$, and hence it has the highest double Sharpe ratio compared to the rest of the portfolios. In this portfolio the allocation is $w_9 = [0,0.942,0,0.0573]^T$, 94.27% for Crew Energy Inc. and 5.73% for N-Viro International Corp.

Table 1. Optimal Portfolio.

		Portfolio Analysis		Bootstrap Resampling		
Portfolio	Yield	Risk	Sharpe Ratio	Standard Deviation	IC 95%	Doble Sharpe Ratio
1	7.4017	42.5770	0.0366	0.1400	-0.1834, 0.3711	0.2615
2	7.7605	42.7579	0.0389	0.1365	-0.1820, 0.3303	0.2849
3	8.1193	43.2959	0.0408	0.1313	-0.2111, 0.3105	0.3107
4	8.4781	44.1780	0.0423	0.1309	-0.2197, 0.2667	0.3231
5	8.8369	45.3867	0.0435	0.1243	-0.2015, 0.2466	0.3499
6	9.1957	47.0668	0.0441	0.1236	-0.2071, 0.2220	0.3569
7	9.5545	49.2696	0.0443	0.1128	-0.2386, 0.2084	0.3927
8	9.9133	51.9286	0.0440	0.1049	-0.2386, 0.1895	0.4196
9	10.2720	54.9776	0.0434	0.0986	-0.1474, 0.1629	0.4397
10	10.6308	58.3555	0.0427	0.1013	-0.0783, 01102	0.4215

Notes: The performance and risk data are annualized. The Sharpe ratio was calculated with the monthly frequency of returns.

In all portfolios, the lower bounds of the 95 percent confidence intervals are negative, suggesting that the performance of the risk-free asset could outperform investments in the stocks of the analyzed companies.

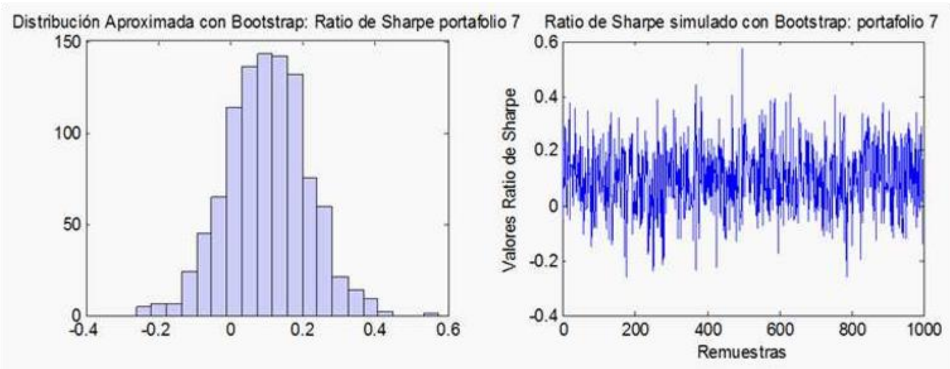

Figure 3. Distribution of the highest Sharpe ratio (portfolio 7) obtained with Bootstrap.

4. Conclusion

In this study, the Sharpe ratio was used as a key metric for evaluating the performance of investment portfolios and determining the optimal composition of portfolios that form the efficient frontier. Since the calculation of the Sharpe ratio involves estimating the expected returns and the standard deviation of the portfolio, which are unknown parameters, therefore there is an intrinsic estimation risk associated with the Sharpe ratio. To address this issue, we applied resampling techniques, specifically bootstrapping, to approximate the value of this estimation risk. This allowed them to construct confidence intervals and double Sharpe ratios, which provided a more robust and accurate picture of the uncertainty associated with the estimation of the Sharpe ratio.

The results of the study showed that the uncertainty in the estimation of the Sharpe ratio can significantly affect the choice of the market portfolio and the ranking of the portfolios along the efficient frontier. In some cases, the results indicated that the performance of the risk-free asset may be equal to or even higher than that of the investments in the shares of the risky portfolio. Based on these findings, there is a clear need for further research to explore ways to reduce the estimation risk associated with the Sharpe ratio and to develop alternative or complementary measures of performance for evaluating investment portfolios. This could include the use of Bayesian methods, machine learning techniques, or other advanced statistical approaches to address the challenges of estimating the unknown parameters required for the calculation of the Sharpe ratio. Additionally, it would be valuable to examine the robustness of the results to different market conditions and to different types of portfolios and investment strategies.

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Declaration of Competing Interest

The author claims that the manuscript is completely original. The author also declares no conflict of interest.

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