Economic Analysis Letters 2024 3(2) 34-47

ANSER PRESS

Economic Analysis Letters

Homepage: https://www.anserpress.org/journal/eal

Vector Error Correction Models with Stationary and Nonstationary Variables

Pu Chen[a,*]

[a]*Melbourne Institute of Technology, Australia*

ABSTRACT

Vector Error Correction Models (VECM) have become a standard tool in empirical economics for analyzing nonstationary time series data because they integrate two key concepts in economics: equilibrium and dynamic adjustment in a single model. The current standard VECM procedure is limited to time series data with the same degree of integration, i.e., all I(1) variables. However, empirical studies often involve time series data with different de-grees of integration, necessitating the simultaneous handling of I(1) and I(0) time series. This paper extends the standard VECM to accommodate mixed I(1) and I(0) variables. The conditions for the mixed VECM are derived, and consequently, we present a test and estimation for the mixed VECM.

KEYWORDS

VECM; Cointegration; Stationary Variables

Corresponding author: Pu Chen
E-mail address: pchen@academic.mit.edu.au

ISSN 2972-3272
doi: 10.58567/eal03020004

Received 10 December 2023; Accepted 14 January 2024; Available online 18 February 2024; Version of Record 15 June 2024

1. Introduction

Vector Error Correction Models (VECM), wherein certain linear combinations of nonstationary variables represent stationary cointegration relations reflecting equilibrium conditions, stand as the most popular method for modeling nonstationary macroeconomic variables. The formulation of a VECM is expressed as follows:

$$\Delta Y_t = \alpha\beta' Y_{t-1} + \sum_{l=1}^{p-1} \Pi_l \Delta Y_{t-l} + U_t \tag{1}$$

Here Y_t is an n dimensional vector, α and β are $n \times h$ dimensional matrices with rank h. $\beta' Y_{t-1} = 0$ represents h equilibrium relations. A deviation from the equilibrium $\beta' Y_{t-1} \neq 0$ will cause the system variable Y_t to adjust by ΔY_t. α is the adjustment coefficient that links the deviation from the equilibrium $\beta' Y_{t-1} \neq 0$ to the system adjustment ΔY_t.

The standard method for building a VECM is as follows: (1) Run unit root tests for each time series in Y_t. (2) If all of the series are I(1), use the Johansen test to determine the cointegration rank h. (3) Use the Johansen procedure to estimate the VECM. This standard approach works well if all Y_t components are $I(1)$ series. In empirical studies, however, not all components of Y_t are $I(1)$ in all cases; some times Y_t may contain both $I(1)$ and $I(0)$ components. In this case, the Johansen procedure cannot be used directly. Cointegration analysis with I(0) and I(1) variables can be treated by ARDL bound test[1]. The bound test, however, is a single equation approach that can only test one cointegration relation. A two-step heuristic approach to modelling both I(1) and I(0) in a VECM is presented, for example, in Hamilton (1994) for a fourvariable system. y_{1t} is stationary. $\mathbf{y}_{2t} = (y_{2t}, y_{3t}, y_{4t})$ the components are each individually I(1). One cointegration relation among the three I(1) variables is concluded in the first step of cointegration analysis, and then a VECM is presented as follows.

$$\begin{aligned}\begin{bmatrix} y_{1,t} \\ \Delta\mathbf{y}_{2,t} \end{bmatrix} &= \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{bmatrix} \zeta_{11}^{(1)} & \zeta_{12}^{(1)} \\ \zeta_{21}^{(1)} & \zeta_{22}^{(1)} \end{bmatrix}\begin{bmatrix} y_{1,t-1} \\ \Delta\mathbf{y}_{2,t-1} \end{bmatrix} + \begin{bmatrix} \zeta_{11}^{(2)} & \zeta_{12}^{(2)} \\ \zeta_{21}^{(2)} & \zeta_{22}^{(2)} \end{bmatrix}\begin{bmatrix} y_{1,t-2} \\ \Delta\mathbf{y}_{2,t-2} \end{bmatrix} + \dots \\ &\dots + \begin{bmatrix} \zeta_{11}^{(p-1)} & \zeta_{12}^{(p-1)} \\ \zeta_{21}^{(p-1)} & \zeta_{22}^{(p-1)} \end{bmatrix}\begin{bmatrix} y_{1,t-p+1} \\ \Delta\mathbf{y}_{2,t-p+1} \end{bmatrix} + \begin{bmatrix} \zeta_{1}^{(0)} \\ \zeta_{2}^{(0)} \end{bmatrix}\mathbf{y}_{2,t-1} + \begin{bmatrix} \epsilon_{1}^{(0)} \\ \epsilon_{2}^{(0)} \end{bmatrix}\end{aligned} \tag{2}$$

where the (3×4) matrix $\begin{bmatrix} \zeta_{1}^{(0)} \\ \zeta_{2}^{(0)} \end{bmatrix}$ is restricted to be $\alpha\beta'$ where α is (4×1) and β is (3×1).

In this paper, we will formalize the procedure proposed in Eq. (2) and present a systemic approach to Vector Error Correction Models (VECM) with mixed I(1) and I(0) variables. It becomes evident that a VECM with mixed I(1) and I(0) variables can be conceptualized as a cointegrated VECM with a set of restrictions in the cointegration space, essentially representing a special case of the conventional cointegrated VECM. Consequently, the Johansen test can be employed to determine the cointegration rank. Moreover, the Johansen procedure can be utilized to test for the presence of I(0) components and to estimate the mixed VECM, as elaborated in the following sections.

2. VECM and the Underlying Process

Eq. (1) presented in the previous section can be reformulated as a vector autoregressive (VAR) model in level of Y_t.

$$Y_t = \sum_{l=1}^{p} \Phi_l Y_{t-l} + \epsilon_t \tag{3}$$

Assumption 1 *The roots of* $|I_n - \sum_{i=1}^{p} \Phi_i z^i| = 0$ *are either outside the unit circle* $|z| = 1$ *or satisfying* $z = 1$.

Assumption 2 *The vector error process* $U_{t=1}^{\infty}$ *is* $IN(0, \Omega)$, Ω *is positive definite.*

[1]See Pesaran et al. (2001) for more details

Eq. (3) with Assumptions 1 and 2 is the same VAR of Eq. (3) and Assumption 1 considered in Johansen (1995) with suppressed deterministic components to simplify the presentation. As a result, the Johansen procedure applies to Eq. (1).

To elucidate our approach to a mixed Vector Error Correction Model (VECM), we initially establish a connection between a cointegrated VECM and an underlying stationary Vector Autoregressive (VAR) process, which will be defined later. We demonstrate that any cointegrated VECM can be viewed as generated from an underlying stationary VAR, and conversely, any stationary VAR can generate a cointegrated VECM that satisfies Assumptions 1 and 2. This foundational principle empowers us to construct a VECM encompassing both I(1) and I(0) components, making it possible to the application of the Johansen procedure. Under Assumption 1 it is shown in Johansen (1995) that $\beta' Y_t$ and $\beta'_\perp \Delta Y_t$ are stationary[2]. Defining $\Delta Z_t = \beta'_\perp \Delta Y_t$ and $X_t = \beta' Y_t$. We call $\begin{bmatrix} \Delta Z_t \\ X_t \end{bmatrix}$ an underlying process of VECM (1) and show in the Appendix that the underlying process $\begin{bmatrix} \Delta Z_t \\ X_t \end{bmatrix}$ is a stationary VAR process.

Definition 1 *For a Eq. (1),* $\begin{bmatrix} \Delta Z_t \\ X_t \end{bmatrix}$ *with* $\Delta Z_t = \beta'_\perp \Delta Y_t$ *and* $X_t = \beta' Y_t$ *is called an underlying process of Eq. (1).*

The following Lemma establishes the connection between a cointegrated VECM and an underlying VAR.

Lemma 1 (VECM and an underlying VAR)

a) For a Eq. (1) satisfying Assumption 1, the underlying process $\begin{bmatrix} \Delta Z_t \\ X_t \end{bmatrix}$ *is a stationary VAR process.*

b) For an n *dimensional stationary VAR, denoting the first* r *components by* ΔZ *and the rest* $n-r$ *components by* X_t*. Let* Y_t *be a full rank linear transformation of* $\begin{bmatrix} Z_t \\ X_t \end{bmatrix}$*:* $Y_t = B \begin{bmatrix} Z_t \\ X_t \end{bmatrix}$*. If* B *is unrestricted* Y_t *is a VAR process satisfying Assumption 1 with* r *unit roots and* $n-r$ *cointegration relations and it can be formulated as Eq. (1).*

Corollary 1 *The Johansen procedure is applicable to a VECM generated from an underlying stationary VAR.*

According to Lemma 1, a cointegrated VECM is always created from an underlying stationary VAR, with the number of unit roots in the VECM being the dimension of Z_t and the number of cointegration relations being the dimension of X_t. How can Y_t have I(0) components? The transformation between Y_t and $\begin{bmatrix} Z_t \\ X_t \end{bmatrix}$ contains the answer to this question. To demonstrate this we decompose B into a 3×3 block:

$$Y_t = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} Z_t \\ X_{1t} \\ X_{2t} \end{bmatrix} \tag{4}$$

where the $n \times n$ matrix B is separated into three blocks each with $r, n-r-k, k$ dimensions with $r+h=n$. B_{11} is $r \times r$, B_{22} is $(n-r-k) \times (n-r-k)$, B_{33} is $k \times k$, and other blocks are defined accordingly. If B_{11}, B_{21}, and B_{31} are nonzero, then all components of Y_t are I(1) because they are linear combinations of Z_t and X_t and Z_t is I(1). If $B_{31} = 0$, the last k components of Y_t are only linear combinations of X_t, implying these k components of Y_t are I(0). As a result, we have a VECM of Y_t with both I(1) and I(0) components. Clearly, $B_{31} = 0$ is both a necessary and sufficient condition for a VECM of Y_t to contain a mix of I(1) and I(0) components. If $B_{21} = 0$ and $B_{31} = 0$ there will be r I(1) and h I(0) components but no cointegration relations. The following lemma summarises the condition for an I(0) and I(1) mixed VECM.

Lemma 2 $B_{31} = 0$ *is a necessary and sufficient condition for a VECM of* Y_t *to contain k I(0) components.*

There is a problem with applying Lemma 2 to determine a mixed VECM because B_{31} is a parameter of the underlying process but not of the corresponding VECM that will be estimated. To turn the conditions in Lemma

[2] In Johansen (1995) the system variable is denoted X_t, while in this paper Y_t is used instead.

2 into a testable hypothesis we must investigate the implications of $B_{31} = 0$ on the parameters of Eq. (1) and determine what restrictions on the parameters of Eq. (1) lead to $B_{31} = 0$.

To that end, we establish a relationship between the transformation matrix B and the parameter in Eq. (1). We consider an underlying VAR of lag 2 without losing generality but simplifying the presentation.

$$\begin{bmatrix} \Delta Z_t \\ X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} \theta_{11}^{(1)} & \theta_{12}^{(1)} & \theta_{13}^{(1)} \\ \theta_{21}^{(1)} & \theta_{22}^{(1)} & \theta_{23}^{(1)} \\ \theta_{31}^{(1)} & \theta_{32}^{(1)} & \theta_{33}^{(1)} \end{bmatrix} \begin{bmatrix} \Delta Z_{t-1} \\ X_{1t-1} \\ X_{2t-1} \end{bmatrix} + \begin{bmatrix} \theta_{11}^{(2)} & \theta_{12}^{(2)} & \theta_{13}^{(2)} \\ \theta_{21}^{(2)} & \theta_{22}^{(2)} & \theta_{23}^{(2)} \\ \theta_{31}^{(2)} & \theta_{32}^{(2)} & \theta_{33}^{(2)} \end{bmatrix} \begin{bmatrix} \Delta Z_{t-2} \\ X_{1t-2} \\ X_{2t-2} \end{bmatrix} + \begin{bmatrix} e_{zt} \\ e_{x1t} \\ e_{x2t} \end{bmatrix} \tag{5}$$

To keep the VECM's lag length at one $\theta_{11}^{(2)}$, $\theta_{21}^{(2)}$, and $\theta_{31}^{(2)}$ are assumed to be zero. Rewriting the underlying Eq. (5) in error correction form we obtain:

$$\begin{bmatrix} \Delta Z_t \\ \Delta X_{1t} \\ \Delta X_{2t} \end{bmatrix} = \begin{bmatrix} 0 & \theta_{12}^{(1)} + \theta_{12}^{(2)} & \theta_{13}^{(1)} + \theta_{13}^{(2)} \\ 0 & \theta_{22}^{(1)} + \theta_{22}^{(2)} - I & \theta_{23}^{(1)} + \theta_{23}^{(2)} \\ 0 & \theta_{32}^{(1)} + \theta_{32}^{(2)} & \theta_{33}^{(1)} + \theta_{33}^{(2)} - I \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ X_{1t-1} \\ X_{2t-1} \end{bmatrix} + \begin{bmatrix} \theta_{11}^{(1)} & -\theta_{12}^{(2)} & -\theta_{13}^{(2)} \\ \theta_{21}^{(1)} & -\theta_{22}^{(2)} & -\theta_{23}^{(2)} \\ \theta_{31}^{(1)} & -\theta_{32}^{(2)} & -\theta_{33}^{(2)} \end{bmatrix} \begin{bmatrix} \Delta Z_{t-1} \\ \Delta X_{1t-1} \\ \Delta X_{2t-1} \end{bmatrix} + \begin{bmatrix} e_{zt} \\ e_{x1t} \\ e_{x2t} \end{bmatrix} \tag{6}$$

Transforming the underling variable $\begin{bmatrix} Z_t \\ X_t \end{bmatrix}$ to Y_t we have:

$$\begin{aligned} &\Delta Y_t \\ &= \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \Delta Z_{t-1} \\ \Delta X_{1t-1} \\ \Delta X_{2t-1} \end{bmatrix} \\ &= \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} 0 & \theta_{12}^{(1)} + \theta_{12}^{(2)} & \theta_{13}^{(1)} + \theta_{13}^{(2)} \\ 0 & \theta_{22}^{(1)} + \theta_{22}^{(2)} - I & \theta_{23}^{(1)} + \theta_{23}^{(2)} \\ 0 & \theta_{32}^{(1)} + \theta_{32}^{(2)} & \theta_{33}^{(1)} + \theta_{33}^{(2)} - I \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}^{-1} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ X_{1t-1} \\ X_{2t-1} \end{bmatrix} \\ &+ \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \theta_{11}^{(1)} & -\theta_{12}^{(2)} & -\theta_{13}^{(2)} \\ \theta_{21}^{(1)} & -\theta_{22}^{(2)} & -\theta_{23}^{(2)} \\ \theta_{31}^{(1)} & -\theta_{32}^{(2)} & -\theta_{33}^{(2)} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}^{-1} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \Delta Z_{t-1} \\ \Delta X_{1t-1} \\ \Delta X_{2t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{bmatrix} \\ &= \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \theta_{12}^{(1)} + \theta_{12}^{(2)} & \theta_{13}^{(1)} + \theta_{13}^{(2)} \\ \theta_{22}^{(1)} + \theta_{22}^{(2)} - I & \theta_{23}^{(1)} + \theta_{23}^{(2)} \\ \theta_{32}^{(1)} + \theta_{32}^{(2)} & \theta_{33}^{(1)} + \theta_{33}^{(2)} - I \end{bmatrix} \begin{bmatrix} B^{21} & B^{22} & B^{23} \\ B^{31} & B^{32} & B^{33} \end{bmatrix} Y_{t-1} \\ &+ \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \theta_{11}^{(1)} & -\theta_{12}^{(2)} & -\theta_{13}^{(2)} \\ \theta_{21}^{(1)} & -\theta_{22}^{(2)} & -\theta_{23}^{(2)} \\ \theta_{31}^{(1)} & -\theta_{32}^{(2)} & -\theta_{33}^{(2)} \end{bmatrix} \begin{bmatrix} B^{11} & B^{12} & B^{13} \\ B^{21} & B^{22} & B^{23} \\ B^{31} & B^{32} & B^{33} \end{bmatrix} \Delta Y_{t-1} + \begin{bmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{bmatrix} \end{aligned}$$

where

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}^{-1} = \begin{bmatrix} B^{11} & B^{12} & B^{13} \\ B^{21} & B^{22} & B^{23} \\ B^{31} & B^{32} & B^{33} \end{bmatrix} \tag{7}$$

We have:

$$\alpha = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \theta_{12}^{(1)} + \theta_{12}^{(2)} & \theta_{13}^{(1)} + \theta_{13}^{(2)} \\ \theta_{22}^{(1)} + \theta_{22}^{(2)} - I & \theta_{23}^{(1)} + \theta_{23}^{(2)} \\ \theta_{32}^{(1)} + \theta_{32}^{(2)} & \theta_{33}^{(1)} + \theta_{33}^{(2)} - I \end{bmatrix} \tag{8}$$

$$\beta' = \begin{bmatrix} B^{21} & B^{22} & B^{23} \\ B^{31} & B^{32} & B^{33} \end{bmatrix} \tag{9}$$

What are the restrictions on β that imply $B_{31} = 0$? It is well known that β is identified up to a full column rank transformation. According to Lemma 7.1 in Johansen (1995), the number of parameters in $\alpha\beta'$ with unconstrained α and β is $n(n-r)+r(n-r)$. To find binding restrictions on β we must first make β exactly identifiable otherwise parameter restrictions and identification restrictions will be mixed up. Let β be identified in the following form.

$\beta' = \begin{bmatrix} B^{21} & I & 0 \\ B^{31} & 0 & I \end{bmatrix}$. By this identification scheme, all elements in α, B^{21}, and B^{31} are free parameters, and the number of these free parameters are $n(n-r)+r(n-r)$.

Since β has full column rank, this identification can be done by premultiplying β' by $\begin{bmatrix} B^{22} & B^{23} \\ B^{32} & B^{33} \end{bmatrix}^{-1}$.

Identifying β' in this way is equivalent to make a full rank transformation of the underlying process of X_t:

$$Y = \underbrace{\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & T_{22} & T_{23} \\ 0 & T_{32} & T_{33} \end{bmatrix}}_{B^*}\underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & T_{22} & T_{23} \\ 0 & T_{32} & T_{33} \end{bmatrix}^{-1}\begin{bmatrix} Z_{t-1} \\ X_{1t-1} \\ X_{2t-1} \end{bmatrix}}_{(Y_t', X_{1t}^{*\prime}, X_{2t}^{*\prime})'} \tag{10}$$

such that the inverse of B^* has the following form:

$$B^* = \begin{bmatrix} B_{11} & B_{12}^* & B_{13}^* \\ B_{21} & B_{22}^* & B_{23}^* \\ B_{31} & B_{32}^* & B_{33}^* \end{bmatrix} = \begin{bmatrix} B^{11*} & B^{12*} & B^{13*} \\ B^{21*} & I & 0 \\ B^{31*} & 0 & I \end{bmatrix}^{-1} \tag{11}$$

What are the constrains on this identified β that implies $B_{31} = 0$? Applying the following formula of partial matrix inverse

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & A^{-1}B(D-CA^{-1}B)^{-1} \\ (D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix} \tag{12}$$

to Eq. (11) and taking $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ as the A block, and $\begin{bmatrix} B^{21*} \\ B^{31*} \end{bmatrix}$ as the B block, we have

$$B_{31} = B^{31*}\left[B^{11*} - [B^{12*}, B^{13*}]\begin{bmatrix} B^{21*} \\ B^{31*} \end{bmatrix}\right]^{-1} \tag{13}$$

and conclude $B^{31*} = 0$ implies $B_{31} = 0$ and vice versa. Inserting this restriction into β we have:

$$\beta' = \begin{bmatrix} B^{21*} & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{14}$$

If a k dimensional I(0) component exists in Y_t, Eq. (14) implies that the $(n-r)$-dimensional I(0) space can be decomposed into two orthogonal subspaces: a k-dimensional I(0) subspace containing the k I(0) variables and an $(n-r-k)$-dimensional cointegration space containing linear combinations of the $(n-k)$ I(1) variables. The k I(0) components do not enter the cointegration space and the cointegrating error terms do not mix with the k I(0) components. We notice that this result keeps unchanged if the k-dimensional I(0) subspace and the $(n-r-k)$-dimensional cointegrating space are subject to a full rank transformation respectively. We summarise this result in the following lemma.

Lemma 3 *The necessary and sufficient condition for a VECM to contain k I(0) variables is the cointegrating matrix β cab be written as:* $\beta' = \begin{bmatrix} B^{21} & B^{22} & 0 \\ 0 & 0 & B^{33} \end{bmatrix}$.

The justification for the two-step method outlined in Eq. (2) is, in part, supported by Lemma 3. It establishes that the I(0) components space and the cointegration space are orthogonal, suggesting that only the three I(1) components are actively involved in the cointegration relations. This justification stems from the fact that the I(0) components are generally not weakly exogenous for the conditional process of the I(1) components. Valid inferences based on a partial system can be drawn only when the conditioning variables are weakly exogenous for the parameters of the partial system[3]. To maintain weak exogeneity, the cointegration relation must not influence the dynamics of the I(0) components, implying $\zeta_1^{(0)} = 0$ in Eq. (2)[4].

3. Generating and Testing mixed VECMs

This section uses two examples to show that 1) a data generating process that satisfies the condition in Lemma 2, $B_{31} = 0$ will generate VECM with mixed I(0) and I(1) variables, and 2) a data generating process that satisfies the condition in Lemma 3, $\beta' = \begin{bmatrix} B^{21} & B^{22} & 0 \\ 0 & 0 & B^{33} \end{bmatrix}$ will also generate a VECM with I(0) and I(0) components.

3.1. Data Generating Process of Mixed VECMs

Example 1

$$\begin{bmatrix} \Delta Z_t \\ X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} \theta_{11}^{(1)} & \theta_{12}^{(1)} & \theta_{13}^{(1)} \\ \theta_{21}^{(1)} & \theta_{22}^{(1)} & \theta_{23}^{(1)} \\ \theta_{31}^{(1)} & \theta_{32}^{(1)} & \theta_{33}^{(1)} \end{bmatrix} \begin{bmatrix} \Delta Z_{t-1} \\ X_{1t-1} \\ X_{2t-1} \end{bmatrix} + \begin{bmatrix} \theta_{11}^{(2)} & \theta_{12}^{(2)} & \theta_{13}^{(2)} \\ \theta_{21}^{(2)} & \theta_{22}^{(2)} & \theta_{23}^{(2)} \\ \theta_{31}^{(2)} & \theta_{32}^{(2)} & \theta_{33}^{(2)} \end{bmatrix} \begin{bmatrix} \Delta Z_{t-2} \\ X_{1t-2} \\ X_{2t-2} \end{bmatrix} + \begin{bmatrix} e_{zt} \\ e_{x1t} \\ e_{x2t} \end{bmatrix} \tag{15}$$

$$Y_t = B \begin{bmatrix} Z_t \\ X_{1t} \\ X_{2t} \end{bmatrix} \tag{16}$$

		$\theta^{(1)}$			
-0.06	*-0.52*	*-0.05*	*0.15*	*-0.05*	*-0.1*
-0.52	*0.25*	*0.28*	*-0.07*	*0.04*	*0.03*
-0.05	*0.28*	*-0.33*	*-0.07*	*0.29*	*-0.2*
0.15	*-0.07*	*-0.07*	*-0.44*	*0.15*	*-0.05*
-0.05	*0.04*	*0.29*	*0.15*	*-0.27*	*-0.44*
-0.1	*0.03*	*-0.2*	*-0.05*	*-0.44*	*0.26*
		$\theta^{(2)}$			
0	*0*	*0.07*	*0.09*	*0*	*0.04*
0	*0*	*0.01*	*0.06*	*-0.04*	*0.01*
0	*0*	*0.03*	*0.04*	*0.08*	*0.1*
0	*0*	*0.04*	*0.01*	*0.04*	*0.06*
0	*0*	*0.08*	*0.04*	*0*	*0.07*
0	*0*	*0.01*	*0.06*	*0.07*	*0.05*
		B			
-0.6	*1.5*	*0.42*	*0.55*	*0.35*	*-1.83*
0.32	*0.15*	*-0.23*	*-0.5*	*-0.66*	*-0.35*
1.87	*-2.61*	*-1.6*	*-0.37*	*-0.05*	*1.22*
-1.77	*-0.11*	*0.73*	*0.2*	*-0.5*	*0.34*
0	*0*	*0.3*	*-0.6*	*0.37*	*0.38*
0	*0*	*-0.4*	*0.5*	*0.23*	*-0.79*

Figure 1 shows one realisation of the simulated data. It is evident from the graphs in Figure 1 that there are 4 I(1) and 2 I(0) variables.

[3] See Engle et al. (1983) for a more detailed explanation.

[4] For further details, refer to Harbo et al. (1998) and Moral-Benito and Servén (2015).

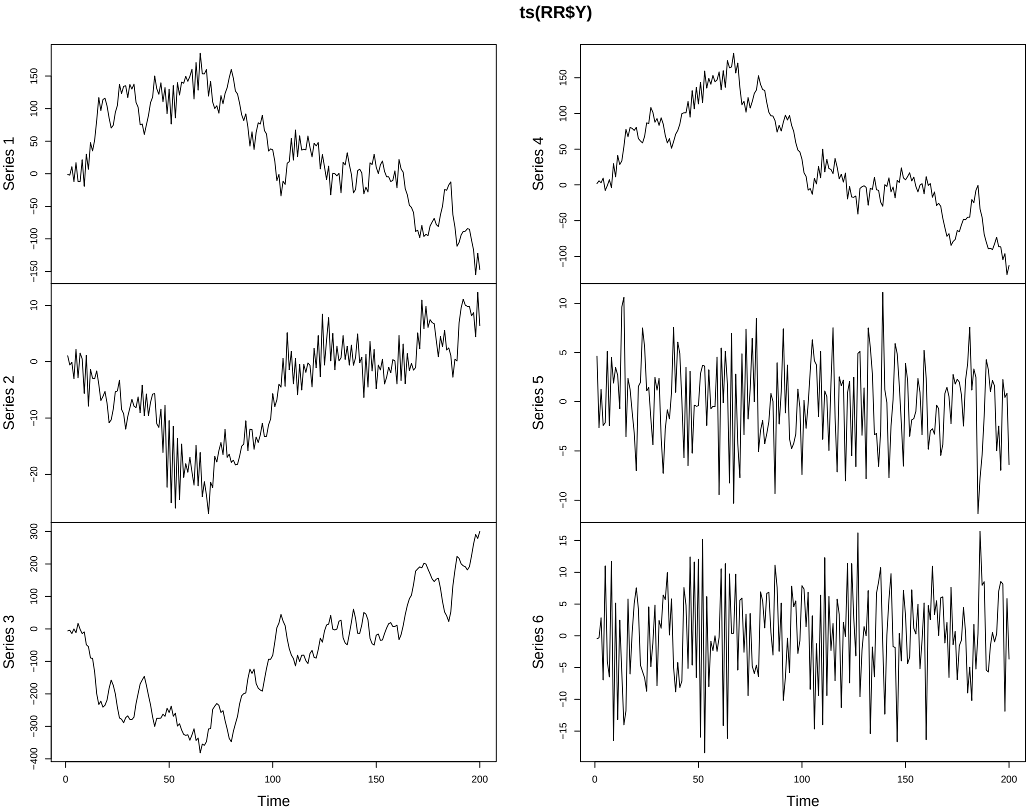

Figure 1. Time series generated from example 1.

Example 2 *The is an example in which data are generated from a VECM with restricted β such that the last two components are I(0).*

$$\Delta Y_t = C + \alpha\beta' Y_{t-1} + \Psi \Delta Y_{t-1} + \epsilon_t \qquad \epsilon_t \sim IN(0, \Omega)$$

α				β				Ψ					
-0.09	0.54	6.05	14.74	1	1	0	0	-1.11	0.71	-0.54	0.04	7.42	-9.35
0.21	0.55	2.78	7.64	-1.34	-3.8	0	0	-0.52	0.4	-0.26	-0.13	4.55	-5.86
0.3	-0.23	-0.11	-1.32	-0.27	0.59	0	0	0.73	-0.28	0.41	-0.33	-4.16	5.16
0.13	0.32	5.32	12.59	-0.29	0.89	0	0	-0.38	0.38	-0.44	-0.12	5.96	-7.33
-0.13	-0.06	0.12	-1.65	0	0	-2.13	-0.21	0.12	-0.13	0.06	0.05	-1.21	1.49
-0.19	0.06	-0.9	-3.39	0	0	1	1	0.26	-0.07	0.22	-0.04	-2.76	3.26

One realization of the simulated data is depicted in Figure 2. The graphs in Figure 2 show clearly that there are 4 I(1) and 2 I(0) variables.

3.2. Testing Mixed VECMs and Parameter Estimation

As discussed in the previous section, VECM with mixed I(0) and I(1) variables can be seen as a special case of a conventional cointegrated VECM where the cointegrating vectors subject to a set of linear restrictions. Therefore the standard Johansen procedure is applicable. We can apply the Johansen procedure to determine the cointegration rank, to test the presence of I(0) variables in the system, and to estimate the parameters of the mixed VECM.

The restrictions on the cointegrating vectors β that leads to a mixed VECM are linear restrictions on β. Following Boswijk and Doornik (2004) linear restrictions can be formulated in the following form.

$$\mathrm{vec}(\alpha') = G\psi, \qquad \mathrm{vec}\beta = H\phi + h_0 \tag{17}$$

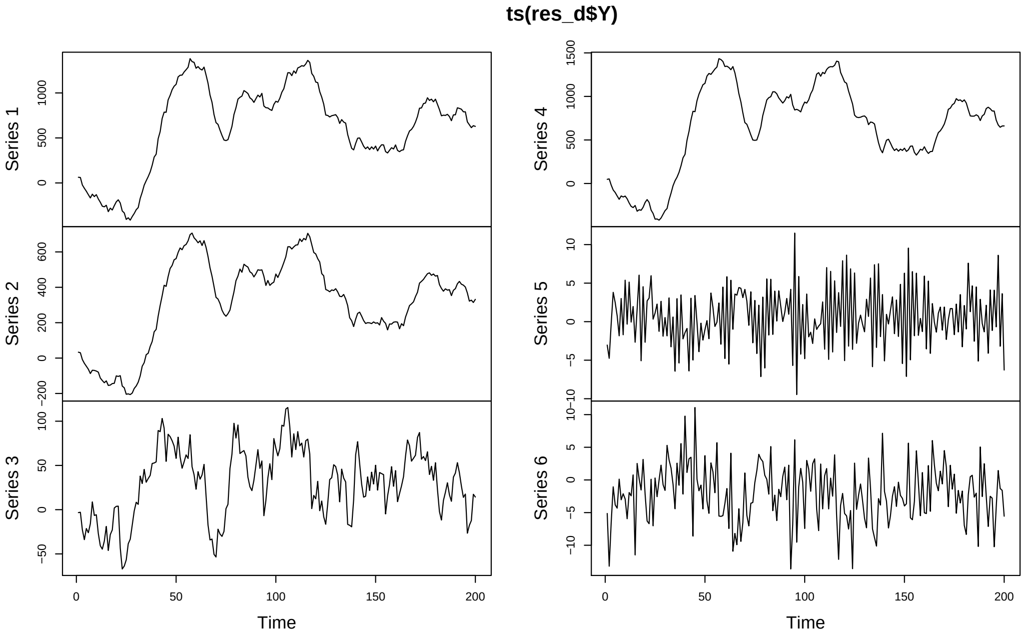

Figure 2. Time series generated from example 2.

Boswijk and Doornik (2004) propose the following iteration process to calculate the constrained parameter and thus to test the hypothesis related to the constrains.

$$\tilde{\psi}(\phi,\Omega)=[G'(\Omega^{-1}\otimes\beta' S_{11}\beta)G]^{-1}(\Omega^{-1}\otimes\beta' S_{11})\text{vec}(\hat{\Pi_{LS}'})$$

$$\tilde{\phi}(\psi,\Omega)=[H'(\alpha'\Omega^{-1}\alpha\otimes S_{11}\beta)H]^{-1}H'(\alpha'\Omega^{-1}\otimes S_{11})\left[\text{vec}\left(\hat{\Pi_{LS}'}\right)-(\alpha\otimes I_{p1})h_0\right]$$

$$\tilde{\Omega}(\psi,\phi)=S_{00}-S_{01}\beta\alpha'-\alpha\beta' S_{10}+\alpha\beta' S_{11}\beta\alpha'$$

Starting from a set of initial values (ψ_0,ϕ_0,Ω_0), the iterations then become $\tilde{\phi}_j=\tilde{\phi}(\psi_{j-1},\Omega_{j-1})$, $\tilde{\psi}_j=\tilde{\psi}(\phi_j,\Omega_{j-1})$, $\tilde{\Omega}_j=\tilde{\Omega}(\phi_j,\Omega_j)$. The iteration procedure stops when the changes in estimated values between two steps are below a prescribed threshold. S_{01}, S_{11}, and S_{00} are the cross products of the two auxiliary regressions in the Johansen procedure. We use this procedure to test the presence of a mixed VECM.

According to Boswijk and Doornik (2004), the following likelihood ratio is asymptotically χ^2 with degree of freedom rk

$$LR=T\left[\log|\tilde{\Omega}|-\log|S_{00}|-\sum_{i=1}^{r}\log\left(1-\hat{\lambda}_i\right)\right]\xrightarrow{d}\chi^2(rk), \quad (18)$$

where $\tilde{\Omega}$ is the constrained estimate obtained from the iteration above and $\log|S_{00}|-\sum_{i=1}^{r}\log(1-\hat{\lambda}_i)$ is the unconstrained estimate from the Johansen procedure.

Examples (continue)

Applying Johansen's test to the data generated in Example 1, we obtain the following result.

```
Johansen test of Example 1
teststatistic critical_value
crk <= 0 |   1287.128961        40.19
crk <= 1 |    631.081580        34.03
crk <= 2 |    145.332672        27.80
crk <= 3 |    113.392495        21.49
crk <= 4 |      8.438725        15.02
crk <= 5 |      1.121140         8.19
```

It is to note that the output of Johansen's test above indicates the dimension of the I(0) space. It is the sum of the independent cointegration relations and the number of I(0) components in the system. For data generated from the parameters in Example 1, we run likelihood ratio tests of one, two three I(0) components, respectively. The corresponding p-value are 0.43, 0.34, and 0.0. We conclude there are two I(0) components in the system. It follows that there are two independent cointegration relations among the 4 I(1) variables.

For the data generated from the parameters in Example 2, we have similar results.

```
Johansen test of Example 2
teststatistic critical_value
crk <= 0 |   1022.555928         40.19
crk <= 1 |    227.575481         34.03
crk <= 2 |     67.163218         27.80
crk <= 3 |     46.358663         21.49
crk <= 4 |      7.232841         15.02
crk <= 5 |      2.674875          8.19
```

Johansen's test results show that the dimension of the I(0) space is 4. The p-values of the likelihood ratio tests of one, two, and three I(0) components are 0.62, 0.48, and 0.0 respectively. We conclude there are two I(0) components in Y_t and two independent cointegration relations among the 4 I(1) variables in the system.

4. An Illustrative Empirical Application

In this section, we provide an example of how a mixed VECM is used in an empirical investigation. We are interested in determining how the unemployment rate (U_t), productivity (yn_t), and capacity utilisation (V_{ct}) affect the real wage (rw_t) in Germany.

The information comes from FRED Economic Data and OECD Statistics. The data spans the quarters of 1990 Q1 and 2021 Q2. Figure 3 contains plots of the time series.

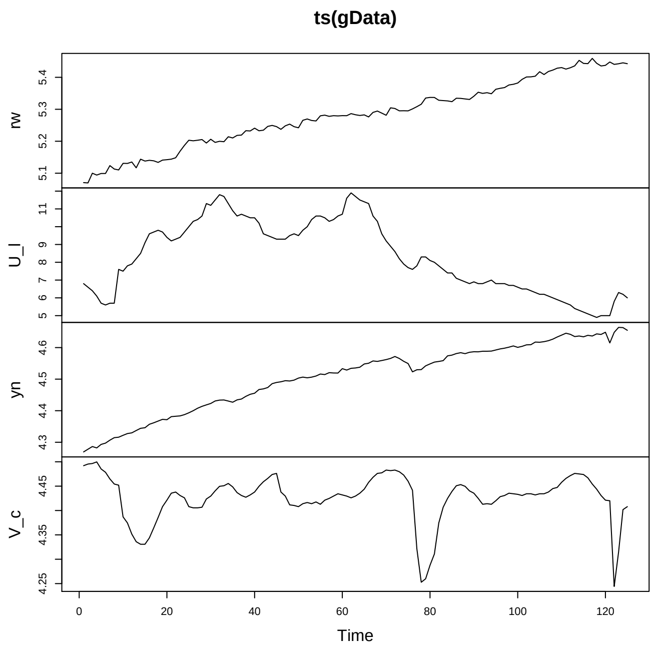

Figure 3. Time series used in the empirical example.

variable	Trnasformation	Description of the untransformed series
w	log(DEULCWRMN01IXOBQ)	German wage index, source: FRED
	- log(DEUCPIALLMINMEIQ)	German CPI, source: FRED
U_l	LMUNRRTTDEM156SQ	German unemployment rate: FRED
V_c	log(BSCURT02DEQ160S)	German capacity utilization rate, source: OECD
yn	DEURGDPHq	Germany, productivity index, source: FRED

Unit root tests of the 4 time series indicate that rw, U_l and yn are I(1) while V_c is I(0). Based on BIC criterion, we specify a VAR(2) for the 4 variables in level. Johansen's test gives the following results.

```
teststatistic critical_value
crk <= 0 |     31.785720          27.80
crk <= 1 |     23.184527          21.49
crk <= 2 |      7.320134          15.02
crk <= 3 |      1.031860           8.19
```

The output above indicates that I(0) space has a dimension of 2. P-values of the likelihood ratio tests of one I(0) components is 8.2%. We conclude that there is one I(0) component and one cointegration relation in the system. The estimated cointegrating vector $\beta = (1, 0.002, -1.20, 0)$ can be reformulated as a long-run real wage equation.

$$\log w_t = -0.002 U_{l,t} + 1.20 \log yn_t \tag{19}$$

This long-term equation shows that productivity has a positive effect on real wages while unemployment has a negative effect. Real wages have an elasticity of 1.2 to productivity, which means real wages will increase 1.2% for every 1% increase in productivity. The elastic response of the real wage to productive growth implies that the wage share in total output is increasing, which contradicts Bowley's law.[5] In contrast, the wage share is "relatively stable" over the long run. Therefore we test the unit elasticity hypothesis. $H_0 : \beta_3 = 1$ $H_1 : \beta_3 > 1$. The likelihood ratio test has a p-value of 4%. The null hypothesis is rejected at the significance level of 5% but not rejected a the level of 1%. Because, despite heated debates, Bowley's is a reliable long-run benchmark (See KRäMER (2011) and Carter (2007) for more detailed discussions.), we choose the significance level at 1% and do not reject the null hypothesis that the elasticity coefficient is 1. As a result, we have a long run real wage equation:

$$\log w_t = -0.007 U_{l,t} + \log yn_t \tag{20}$$

This equation says if the unemployment rate increases by 1%, the real wage will decrease by 0.7%. A deviation from this long run relation is measure by the error term $\beta' Y_{t-1}$ which is depicted in the following diagram.

The plot in Figure 4 shows the real wage can deviate from the long run equilibrium level by about $\pm 5\%$. The cointegration relationship also shows that the rate of capacity utilisation has no long-run effect on the real wage. However, it may have a short-term impact on the real wage. To evaluate the short run dynamics, we calculate impulse response functions of the mixed VECM shown in Figure 5.

In this set of impulse response functions, the Cholezky decomposition is used in the order (yn, Ul, rw, Vc). The shocks are one unit of the corresponding variables. A positive productivity shock, according to the impulse response function, reduces unemployment rate, raises the real wage, and increases capacity utilisation. A one standard deviation unemployment shock has no effect on productivity, but it lowers the real wage and temporarily reduces capacity utilisation.

5. Concluding Remarks

In this paper, we extend the conventional cointegrated VECM to mixed cointegrated VECMs that can accommodate I(0) and I(1) variables. A testable necessary and sufficient condition for a mixed cointegrated VECM is

[5] See Hamilton (1994) for more details.

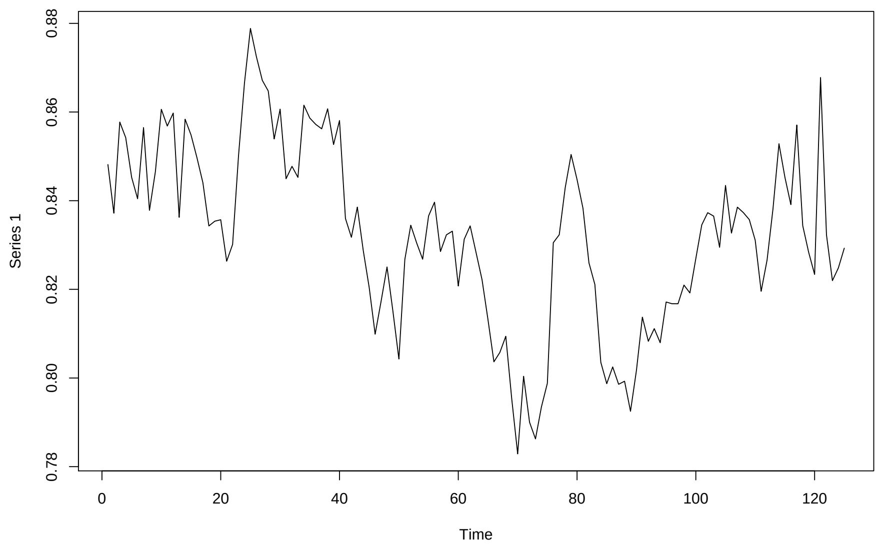

Figure 4. Time series generated from example.

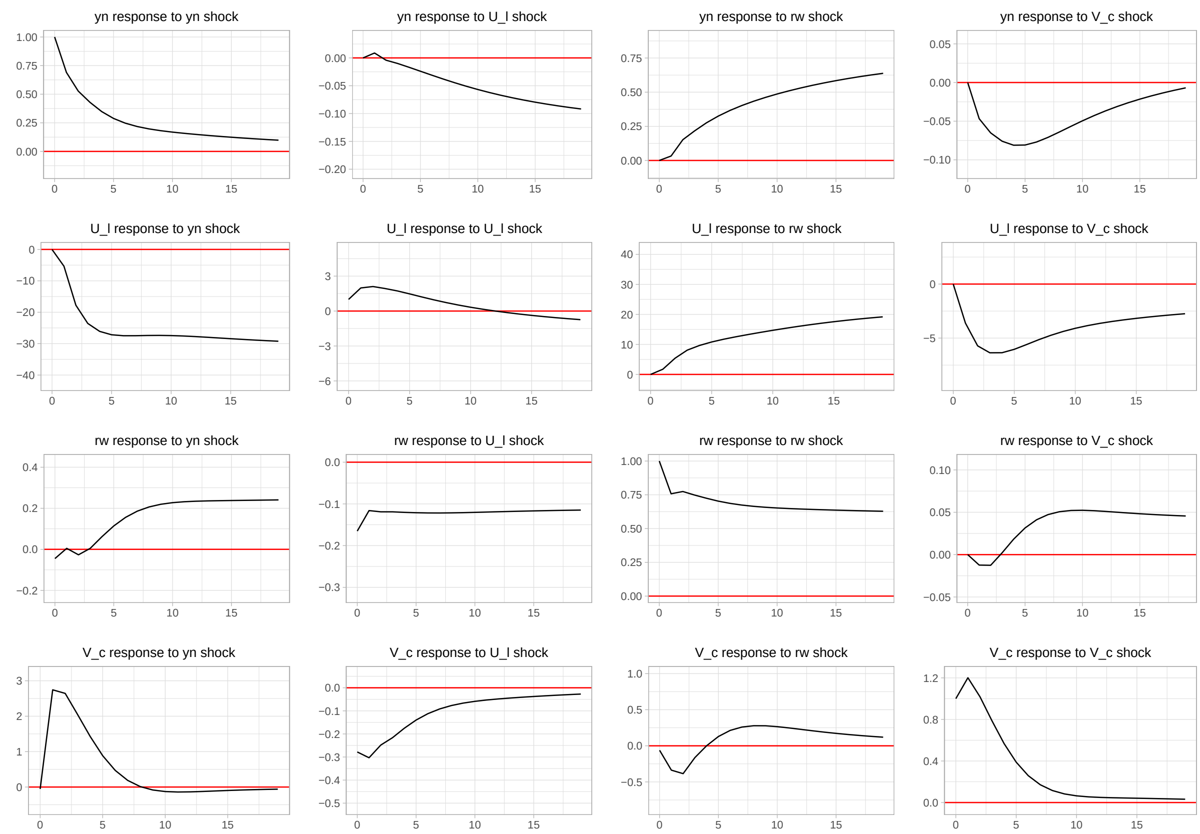

Figure 5. Time series generated from example.

provided. We show that a mixed cointegrated VECM is a special case of a conventional cointegrated VECM. As a result, Johansen's test can be used to test the cointegration rank as well as the constraints on the cointegrating vectors that make a VECM a mixed cointegrated VECM. Practical implication of the results in this paper is that the exiting econometric software packages such as R, RATS, and EVIEWS that can be used to test restrictions on β in a VECM can be directly use to test a mixed VECM and estimate the parameters of the mixed VECM.

Funding Statement

This research received no external funding.

Acknowledgments

Acknowledgments to anonymous referees' comments and editor's effort.

Conflict of interest

All the authors claim that the manuscript is completely original. The author also declare no conflict of interest.

Appendix

Proof of Lemma 1 a)

To simplify the presentation, the lag length p is set to 2 and the deterministic components are suppressed in the following proof without loss of generality. This results in the following VECM

$$\Delta Y_t = \alpha\beta' Y_{t-1} + \Theta \Delta y_{t-1} + u_t$$

Premultiply the equation above by $\begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix}$ we obtain:

$$\begin{aligned}
\begin{bmatrix} \beta'_\perp \Delta Y_t \\ \beta' Y_t - \beta' Y_{t-1} \end{bmatrix} &= \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix} \alpha\beta' Y_{t-1} + \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix} \Theta \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix}^{-1} \begin{bmatrix} \beta'_\perp \Delta Y_{t-1} \\ \beta' Y_{t-1} - \beta' Y_{t-2} \end{bmatrix} + \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix} u_t \\
\begin{bmatrix} \beta'_\perp \Delta Y_t \\ \beta' Y_t \end{bmatrix} &= \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix} \left[\begin{bmatrix} 0 & \alpha \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \Theta \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix}^{-1} \right] \begin{bmatrix} \beta'_\perp \Delta Y_{t-1} \\ \beta' Y_{t-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix} \Theta \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \beta' Y_{t-2} \end{bmatrix} + \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix} u_t
\end{aligned}$$

Replacing $\beta_\perp \Delta Y_t$ and $\beta' Y_t$ by ΔZ_t and X_t respectively, we obtain

$$\begin{aligned}
\begin{bmatrix} \Delta Z_t \\ X_t \end{bmatrix} &= \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix} \left[\begin{bmatrix} 0 & \alpha \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \Theta \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix}^{-1} \right] \begin{bmatrix} \Delta Z_{t-1} \\ X_{t-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix} \Theta \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Delta Z_{t-2} \\ X_{t-2} \end{bmatrix} + \begin{bmatrix} \beta'_\perp \\ \beta' \end{bmatrix} u_t
\end{aligned}$$

The equation above shows the underlying process is a VAR process. Following Theorem 4.2 in Johansen (1995) $\Delta Z = \beta_\perp \Delta Y_t$ and $X = \beta' Y_t$ are stationary, hence the underlying process is a stationary VAR process. This proves Lemma 1 a).

To prove Lemma 1 b) we consider the following stationary VAR. Again, we choose $p = 2$ and suppress the deterministic components.

$$\begin{bmatrix} \Delta Z_t \\ X_t \end{bmatrix} = \zeta_1 \begin{bmatrix} \Delta Z_{t-1} \\ X_{t-1} \end{bmatrix} + \zeta_2 \begin{bmatrix} \Delta Z_{t-2} \\ X_{t-2} \end{bmatrix} + v_t \tag{A1}$$

To keep the lag length of VECM being 1, we assume in addition the coefficient of ΔZ_{t-2} is zero: $\zeta_1 = \begin{bmatrix} \zeta_{11}^{(1)} & \zeta_{12}^{(1)} \\ \zeta_{21}^{(1)} & \zeta_{22}^{(1)} \end{bmatrix}$ $\zeta_2 = \begin{bmatrix} 0 & \zeta_{12}^{(2)} \\ 0 & \zeta_{22}^{(2)} \end{bmatrix}$ Rearrange Eq. (A1):

$$\begin{bmatrix} \Delta Z_t \\ \Delta X_t \end{bmatrix} = \begin{bmatrix} 0 & \zeta_{12}^{(1)} \\ 0 & \zeta_{22}^{(1)} + \zeta_{22}^{(2)} - I \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ X_{t-1} \end{bmatrix} + \begin{bmatrix} \zeta_{11}^{(1)} & \zeta_{12}^{(2)} \\ \zeta_{21}^{(1)} & -\zeta_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \Delta Z_{t-1} \\ \Delta X_{t-1} \end{bmatrix} + v_t \tag{A2}$$

The stationarity assumption of $\begin{bmatrix} \Delta Z_t \\ X_t \end{bmatrix}$ implies the roots of the following characteristic polynomial equation

$$\left|I - \zeta_1 z - \zeta_2 z^2\right| = 0 \tag{A3}$$

lie outside the unit circle $|z| = 1$. Using lag operator the stationary Eq. (A1) can be written as:

$$(I - \zeta_1 L - \zeta_2 L^2)\begin{bmatrix} \Delta Z_t \\ X_t \end{bmatrix} = 0$$

Inserting $\begin{bmatrix} \Delta Z_t \\ X_t \end{bmatrix} = \begin{bmatrix} I - L & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Z_t \\ X_t \end{bmatrix}$ into the equation above we have

$$(I - \zeta_1 L - \zeta_2 L^2)\begin{bmatrix} I - L & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Z_t \\ X_t \end{bmatrix} = 0 \tag{A4}$$

Obviously

$$\left|(I - \zeta_1 z - \zeta_2 z^2)\begin{bmatrix} I(1-z) & 0 \\ 0 & I \end{bmatrix}\right| = \left|(I - \zeta_1 z - \zeta_2 z^2)\right|\left|\begin{bmatrix} I(1-z) & 0 \\ 0 & I \end{bmatrix}\right| = 0$$

has only roots outside the unit circle and unit roots. Hence the VAR of $\begin{bmatrix} Z_t \\ X_t \end{bmatrix}$ satisfies Assumption 1. Next we show a full rank transformation of a VAR process will not change the roots of the characteristic polynomials of the respective VAR processes. For a VAR process

$$X_t = \sum_{l=1}^{p} \Phi_l X_{t-l} + \epsilon_t \tag{A5}$$

the roots the characteristic polynomial of Eq. (A5) is defined by the following equation

$$|I_n - \sum_{i=1}^{p} \Phi_i z^i| = 0 \tag{A6}$$

For a full rank transformation of $W_t = BX_t$ we have

$$W_t = \sum_{l=1}^{p} B\Phi_l B^{-1} W_{t-l} + B\epsilon_t \tag{A7}$$

and its roots is defined by

$$|I_n - \sum_{i=1}^{p} B\Phi_i B^{-1} z^i| = |B(I_n - \sum_{i=1}^{p} \Phi_i z^i)B^{-1}| = 0 \tag{A8}$$

Because B has full rank, the roots of equation (A8) is identical to the roots of equation (A6). Using this results and that Y_t is a full rank transformation of $\begin{bmatrix} Z_t \\ X_t \end{bmatrix}$, we conclude the VAR of Y_t satisfies Assumption 1. This proves Lemma 1 b). □

Proof of Lemma 2

According to the relation between Y_t and the underlying process in Eq. (4), if $B_{31} = 0$, the last k components of Y_t are linear combinations of I(0) variables, these k components are I(0). This proves the sufficiency. If the last k components of Y_t are $I(0)$, the last component of Y_t is I(0). From Eq. (4) we have

$$Y_{n,t} = \sum_{i=1}^{r} b_{n,i} Z_{i,t} + \sum_{j=1}^{n-r} b_{n,j+r} X_{j,t} \tag{A9}$$

Because $Z_{i,t}$ for $i = 1, 2, ..., r$ are independent $I(1)$ variables, $\sum_{i=1}^{r} b_{n,i} Z_{i,t}$ is an I(1) variable unless $b_{n,i} = 0$ for $i = 1, 2, ..., r$. Because $Y_{n,t}$ is I(0) by assumption it follows $b_{n,i} = 0$ for $i = 1, 2, ..., r$, i.e. the last row in B_{31} is zero. In the same way we can prove each row in B_{31} is zero. This proves the necessity. □

Proof of Lemma 3

To prove the sufficiency, we can premultiply $\beta' = \begin{bmatrix} B^{21} & B^{22} & 0 \\ 0 & 0 & B^{33} \end{bmatrix}$ by $\begin{bmatrix} B^{22} & 0 \\ 0 & B^{33} \end{bmatrix}^{-1}$ and obtain

$$\beta^{*\prime} = \begin{bmatrix} (B^{22})^{-1} B^{21} & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{A10}$$

By the formula of matrix inverse in block form Eq. (12) and Eq. (13) we know $\beta' = \begin{bmatrix} B^{21} & B^{22} & 0 \\ 0 & 0 & B^{33} \end{bmatrix}$ implies $B_{31} = 0$.

To prove the necessity, we assume without loss of generality that the last k components of Y_t are I(0). Then a cointegrating matrix β^* can be written as:

$$\beta^{*\prime} = \begin{bmatrix} B^{*21} & B^{*22} & B^{*23} \\ 0 & 0 & B^{33} \end{bmatrix} \tag{A11}$$

because a full rank transformation of the last k I(0) components of Y_t are still I(0), thus the $n \times k$ matrix $(0, 0, B^{33})$ is a cointegrating matrix of Y_t, containing k independent I(0) components. Premultiply Eq. (A11) by $\begin{bmatrix} I & -B^{23*}(B^{33})^{-1} \\ 0 & I \end{bmatrix}$.

We obtain

$$\beta' = \begin{bmatrix} B^{21} & B^{22} & 0 \\ 0 & 0 & B^{33} \end{bmatrix} \tag{A12}$$

This proves the necessity. It is to note that making $B^{23} = 0$ is not the consequence of k I(0) components in Y_t but the convention of normalization of orthogonal cointegration relations and the I(0) components. □

References

Boswijk, H. P., and Doornik, J. A. (2004). Identifying, estimating and testing restricted cointegrated systems: An overview. *Semantic Scholar* 58(4), 440–465. https://doi.org/10.1111/j.1467-9574.2004.00270.x.

Carter, S. (2007). Real wage productivity elasticity across advanced economies, 1963-1996. *Journal of Post Keynesian Economics* 29(4), 573–600. https://doi.org/10.2753/PKE0160-3477290403.

Engle, R., Hendry, D. F., and Richard, J.-F. (1983). Exogeneity. *Econometrica* 51, 277–304. https://doi.org/10.2307/1911990.

Hamilton, J. D. (1994). *Time Series Analysis*. Princeton University Press.

Harbo, I., Johansen, S., Nielsen, B., and Rahbek, A. (1998). Asymptotic inference on cointegrating rank in partial systems. *Journal of Business & Economic Statistics* 16(4), 388–399. https://doi.org/10.1080/07350015.1998.10524779.

Johansen, S. (1995). *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models*. Oxford University Press.

KRäMER, H. M. (2011). Bowley’s law: The diffusion of an empirical supposition into economic theory. *Papers in Political Economy* 61(2), 19–49. https://doi.org/10.3917/cep.061.0019.

Moral-Benito, E., and Servén, L. (2015). Testing weak exogeneity in cointegrated panels. *Applied Economics* 47(30), 3216–3228. https://doi.org/10.1080/00036846.2015.1013611.

Pesaran, M. H., Shin, Y., and Smith, R. J. (2001). Bounds testing approaches to the analysis of level relationships. *Journal of Applied Econometrics* 16(3), 289–326. https://doi.org/10.1002/jae.616.