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Money holding and budget deficit in a growing economy with consumers living forever

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ABSTRACT

I examine the problem of budget deficit in a growing economy in which consumers hold money as a part of their savings in the case where consumers live forever. For simplicity and tractability I use a discrete time dynamic model and Lagrange multiplier method. In the appendix I briefly explain the solution using a discrete time version of the Hamiltonian method. I will show the following results. 1) Budget deficit is necessary for full employment under constant prices. 2) Inflation is induced if the actual budget deficit is greater than the value at which full employment is achieved under constant prices. 3) If the actual budget deficit is smaller than the value which is necessary and sufficient for full employment under constant prices, a recession occurs. Therefore, balanced budget cannot achieve full employment under constant prices. I do not assume that budget deficit must later be made up by budget surplus.

KEYWORDS

Budget deficit; Growing economy; Infinitely living consumers

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1. Introduction

This paper is an attempt to present a simple theoretical model for the following statement by J.M. Keynes.

"Unemployment develops, that is to say, because people want the moon; — men cannot be employed when the object of desire (i.e. money) is something which cannot be produced and the demand for which cannot be readily choked off. There is no remedy but to persuade the public that green cheese is practically the same thing and to have a green cheese factory (i.e. a central bank) under public control." (Keynes(1936), Chap. 17)

In Japan and many other countries, fiscal deficits and accumulated government debt have become a problem, and it is argued that fiscal soundness must be improved. Is this really the case? Even if the goal of achieving a balanced budget will worsen the economy, impede growth, and create unemployment, does it still make sense to improve fiscal soundness? These questions are the starting point for this study.

In recent some researches I have examined the role of the budget deficits in a growing economy when consumers get utility from money holding as well as consumptions of goods. In these studies I have used an overlapping generations model in which consumers live over two periods, the younger period and the older period. They work in the younger period and retire in the older period. They consume goods in the older period by the savings carried over from the younger period. About the overlapping generations model I referred to Diamond (1965), J. Tanaka (2010, 2011a, 2011b, 2013) and Otaki (2007, 2009, 2015). Such a model may be reasonably realistic, but it may not be very general. In this study, therefore, I would like to examine the same issue under the assumption that people will live forever. I assume the exogenous growth by population growth[1]. I do not assume that a new generation is born, but rather that the population of the current generation multiplies in the next period. About a model in which consumers infinitely live I refer to Weil (1987, 1989). However, although he used a continuous time model, I use a more tractable discrete time model. My another reference is Tachibana (2006).

The main results of this paper are as follows. (1) We need budget deficit for full employment under constant price in a growing economy. (2) If the actual budget deficit is greater (smaller) than the value which is necessary and sufficient for full employment under constant price, an inflation (a recession) occurs. Balanced budget cannot achieve full employment under constant price. We should not assume that budget deficit must later be made up by budget surplus. There is a persistent belief that accumulated government debt must eventually be repaid by running budget surpluses, but such a move would lead to a severe recession. It should be recognized that even if a certain level of budget deficit continues under steady-state conditions, it will not cause inflation unless it is excessive, and that it is essential to maintain full employment at stable prices[2].

In the next section, I present the model of this paper and prove the above main results. In Section 3 I calculate the explicit values of the savings and money holding[3]. In Section 4 I present a brief numerical example.

The proofs of each proposition are largely contained in the appendices.

2. Money holding and budget deficit in a growing economy

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[1] In other studies I used an endogenous growth model according to Grossman and Yanagawa(1993) and Maebayashi and J. Tanaka(2021).

[2] The following discussion focuses on the steady state without inflation, but this does not mean assuming price rigidity, but rather examining the government's fiscal policy to achieve and maintain full employment while preventing inflation from occurring. Depending on the size of the budget deficit, full employment with inflation may be considered.

[3] In this paper, I assume that budget deficits are financed by the issuance of money, not government bonds. The issuance of money is not more inflationary than government bonds. If government bonds are issued in place of money with the same liquidity as money, the interest on these bonds would add to the budget deficit, thus narrowing the scope for fiscal spending.

2.1. Consumers' behavior

I introduce money demand or money holding by consumers into a simple exogenous growth model in which consumers infinitely live. Consumers use wages and profits earned from firms and savings left over from the previous period to consume a homogeneous good in each period and save for the next period. The savings include capital as well as money holding. The consumer's utility depends on the consumption and the real money holding. The population of consumers grows by constant rate from a period to the next period. This does not mean that a new generation will be born, but rather that the population of the current generation will multiply at a constant rate.

The consumer's utility over an infinite period of time is expressed as follows.

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+\delta}\right)^{t} u\left(c_t,\frac{m_{t+1}}{p_t}\right),\qquad \delta>0,\qquad t=0,1,\qquad \cdots,\infty. \tag{1}$$

Specifically, the utility function is

$$(c_t,\frac{m_{t+1}}{p_t})=\alpha\ln c_t+(1-\alpha)\ln\frac{m_{t+1}}{p_t},\qquad 0<\alpha<1.$$

c_t and p_t are the (real value of) consumption per capita and the price of the good in Period t. m_{t+1} is the nominal value of the money holding of the consumer at the end of Period t. It will be carried over to the next period. Therefore, $\frac{m_{t+1}}{p_t}$ is the real value of the money holding. δ is the discount rate. α is the parameter of the utility function. Let s_{t+1} be the savings of the consumer at the end of Period t, and b_{t+1} be its value in Period $t+1$.

Then, we have

$$b_{t+1}=(1+r_{t+1})(s_{t+1}-m_{t+1})+m_{t+1}. \tag{2}$$

$s_{t+1}-m_{t+1}$ represents the portion of savings that is invested in productive capital, which generates interest in the next period. r_{t+1} is the interest rate of the capital in Period $t+1$. Similarly,

$$b_t=(1+r_t)(s_t-m_t)+m_t.$$

From (2),

$$s_{t+1}=\frac{b_{t+1}-m_{t+1}}{1+r_{t+1}}+m_{t+1}=\frac{b_{t+1}}{1+r_{t+1}}+\frac{r_{t+1}}{1+r_{t+1}}m_{t+1}.$$

The budget constraint for the consumer in period t is

$$p_tc_t+s_{t+1}=(1-\tau)w_tl_t+\frac{b_t}{1+n},\qquad t=0,1,\qquad \cdots,\infty.$$

w_t is the nominal wage rate, and l_t is the indicator that represents whether the consumer is employed or not. τ is the tax rate, $0<\tau<1$. If we assume full employment, $l_t=1$ for all consumers. $n>0$ is the population growth rate. The savings at the end of Period t are distributed equally among consumers in Period $t+1$. From this, we obtain

$$p_tc_t+\frac{1}{1+r_{t+1}}(b_{t+1}-m_{t+1})+m_{t+1}=(1-\tau)w_tl_t+\frac{b_t}{1+n},$$

and

$$p_t c_t + \frac{r_{t+1}}{1+r_{t+1}} m_{t+1} = (1-\tau) w_t l_t + \frac{b_t}{1+n} - \frac{1}{1+r_{t+1}} b_{t+1}, \quad t = 0,1, \quad \cdots, \infty. \tag{3}$$

It is rewritten as

$$b_{t+1} = (1+r_{t+1})(1-\tau) w_t l_t - (1+r_{t+1}) p_t c_t - r_{t+1} m_{t+1} + \frac{1+r_{t+1}}{1+n} b_t, \qquad t = 0,1, \qquad \cdots, \infty.$$

Further,

$$b_{t+1} - b_t = (1+r_{t+1})(1-\tau) w_t l_t - (1+r_{t+1}) p_t c_t - r_{t+1} m_{t+1} + \frac{r_{t+1} - n}{1+n} b_t.$$

By Lagrangian method we obtain the following consumption and money holding of the consumers (please see Appendix 1).

$$c_t = \frac{\alpha}{p_t}\left[(1-\tau) w_t l_t + \frac{b_t}{1+n} - \frac{1}{1+r_{t+1}} b_{t+1}\right], \tag{4}$$

and

$$m_{t+1} = (1-\alpha)\frac{1+r_{t+1}}{r_{t+1}}\left[(1-\tau) w_t l_t + \frac{b_t}{1+n} - \frac{1}{1+r_{t+1}} b_{t+1}\right]. \tag{5}$$

Let L_t^f be labor supply or population in Period t. Under full employment the employment equals L_t^f. Since it grows at the rate of n,

$$L_t^f = (1+n) L_{t-1}^f.$$

The total consumption in Period t is

$$C_t = \frac{\alpha}{p_t}\left[(1-\tau) w_t L_t + \frac{b_t}{1+n} L_t^f - \frac{1}{1+r_{t+1}} B_{t+1}\right].$$

L_t is the employment in Period t. B_{t+1} is the value of the total savings in Period $t+1$. The total nominal consumption is

$$p_t C_t = \alpha\left[(1-\tau) w_t L_t + \frac{b_t}{1+n} L_t^f - \frac{1}{1+r_{t+1}} B_{t+1}\right].$$

On the other hand, the total nominal money holding is

$$M_{t+1} = (1-\alpha)\frac{1+r_{t+1}}{r_{t+1}}\left[(1-\tau) w_t L_t + \frac{b_t}{1+n} L_t^f - \frac{1}{1+r_{t+1}} B_{t+1}\right].$$

The total nominal savings is

$$S_{t+1} = \frac{B_{t+1} - M_{t+1}}{1+r_{t+1}} + M_{r+1} = \frac{B_{t+1}}{1+r_{t+1}} + \frac{r_{t+1}}{1+r_{t+1}} M_{r+1} = \frac{\alpha B_{t+1}}{1+r_{t+1}} + (1-\alpha)\left[(1-\tau) w_t L_t + \frac{b_t}{1+n} L_t^f\right].$$

The real value of the capital in Period $t+1$ is

$$K_{t+1} = \frac{1}{(1+r_{t+1}) p_t}(B_{t+1} - M_{t+1}) = \frac{1}{p_t}(S_{t+1} - M_{t+1}).$$

Similarly, the real value of the capital in Period t is

$$K_t = \frac{1}{(1+r_t)p_{t-1}}(B_t - M_t) = \frac{1}{p_{t-1}}(S_t - M_t).$$

The relationships between the total values of the variables and the per capita values are

$$B_{t+1} = b_{t+1}L_t^f, \qquad M_{t+1} = m_{t+1}L_t^f, \qquad S_{t+1} = s_{t+1}L_t^f.$$

Similarly, for the values in Period t, we have

$$B_t = b_t L_{t-1}^f, \qquad M_t = m_t L_{t-1}^f, \qquad S_t = s_t L_{t-1}^f.$$

2.2. Firms' behavior

Let y_t be the output, K_t be the capital, L_t be the employment of firms in Period t. Then, the production function is written as follows.

$$y_t = F(K_t, L_t) = L_t f(k_t) = L_t F(k_t, 1).$$

I assume the constant returns to scale property for the production function. k_t is the real capital per labor. The number of firms is normalized to one. Each firm maximizes its profit in each period. The profit of a firm is

$$\pi_t = p_t y_t - r_t K_t - w_t L_t = p_t L_t f(k_t) - p_t r_t K_t - w_t L_t.$$

The first order conditions for profit maximization are

$$p_t r_t = p_t \frac{\partial F}{\partial K_t} = p_t f'(k_t),$$

and

$$w_t = p_t \frac{\partial F}{\partial L_t} = p_t[f(k_t) - f'(k_t)k_t].$$

$\frac{\partial F}{\partial K_t}$ and $\frac{\partial F}{\partial L_t}$ are the marginal productivity of capital and that of labor. From them, we have

$$w_t L_t = p_t\,[f(k_t) - f'(k_t)k_t]L_t,$$

and

$$p_t r_t K_t = p_t f'(k_t) K_t = p_t f'(k_t) k_t L_t.$$

Thus, we obtain

$$p_y y_t = w_t L_t + p_t r_t K_t.$$

This is the total nominal supply of the good.

2.3. Market equilibrium

In this subsection I will show the main result of this paper. The total nominal consumption demand is

$$p_tC_t = \alpha\left[(1-\tau)w_tL_t + B_t - \frac{1}{1+r_{t+1}}B_{t+1}\right].$$

Let G_t be the fiscal expenditure in Period t. The total nominal demand is

$$G_t + \alpha\left[(1-\tau)w_tL_t + B_t - \tfrac{1}{1+r_{t+1}}B_{t+1}\right] + p_t(K_{t+1} - K_t).$$

The market clearing condition is

$$G_t + \alpha\left[(1-\tau)w_tL_t + B_t - \frac{1}{1+r_{t+1}}B_{t+1}\right] + p_t(K_{t+1} - K_t) = w_tL_t + p_tr_tK_t. \tag{6}$$

$p_t(K_{t+1} - K_t)$ is the investment in Period t which is the nominal value of the increase in the capital from Period t to Period t+1.

I will show the following proposition. Please see Appendix 2 for the proof of this proposition.

Proposition 1

In a growing economy, if consumers get utility from money holding, in the steady state under full employment with constant price, we need positive budget deficit.

Proof: Appendix 2.

The main equations in the proof are

$$G_t - \tau w_tL_t = \frac{p_t}{p_{t-1}}(B_t - M_t) + M_{t+1} - B_t, \tag{A.7}$$

and, with $p_t = p_{t-1}$,

$$G_t - \tau w_tL_t^f, = M_{t+1} - M_t = nM_t. \tag{A.8}$$

Taking M_t as given, if full employment has been achieved until Period $t-1$, then it is necessary and sufficient to create the budget deficit shown in (A.8) to achieve inflation-free full employment in Period t.

Inflation and recession

About inflation and recession we can get the following results.

Proposition 2

If the budget deficit is larger than the value which is necessary and sufficient to achieve full employment under constant price, inflation is triggered. On the other hand, if the budget deficit is smaller than the value which is necessary and sufficient to achieve full employment under constant price, a recession occurs.

Proof:

If the actual budget deficit is larger than the value which is necessary and sufficient for full employment under constant price (expressed by (A.7) in Appendix 2 with $p_t = p_{t-1}$, or (A.8)), p_t should be larger. Therefore, inflation is triggered. On the other hand, if the actual budget deficit is smaller than that value, M_{t+1} should be smaller. This will be realized through a decline in income and production. (Q.E.D.)

3. Explicit values of the savings and money holding

The equations of individual consumption and money holding expressed in (4) and (5) include savings and are not in closed form. Therefore, I consider the explicit values of some variables, savings and money holding, in this section. I will show the following proposition.

Proposition 3

The explicit solutions of the value of the savings in Period t and the value of the money holding at the end of Period t are

$$B_t = \frac{(1-\alpha)\frac{1+\tilde{r}}{\tilde{r}}(1-\tau)w_t L_t^f + (1+\tilde{r})(1+\mathrm{n})p_t\widetilde{K}_t}{1+n-(1-\alpha)\frac{\tilde{r}-n}{\tilde{r}}}, \tag{7}$$

and

$$M_{t+1} = (1+n)\frac{(1-\alpha)\frac{1+\tilde{r}}{\tilde{r}}(1-\tau)w_t L_t^f + (1+\tilde{r})(1+n)p_t\widetilde{K}_t}{1+n-(1-\alpha)\frac{\tilde{r}-n}{\tilde{r}}} - (1+\tilde{r})(1+n)p_t$$
$$= (1+n)(1-\alpha)\frac{\frac{1+\tilde{r}}{\tilde{r}}(1-\tau)w_t L_t^f + (1+\tilde{r})\frac{\tilde{r}-n}{\tilde{r}}p_t\widetilde{K}_t}{1+n-(1-\alpha)\frac{\tilde{r}-n}{\tilde{r}}},$$

with

$$\widetilde{K}_t = \tilde{k}L_t^f.$$

$\tilde{k}$ is the equilibrium value of the capital-labor ratio. It is determined by the growth rate and the discount rate. *Proof:* Appendix 3.

4. Numerical example

This section presents a numerical example. Consider the production function expressed as follows;

$$F(L_t, K_t) = L_t^{\frac{1}{2}}K_t^{\frac{1}{2}}, f(k_t) = k_t^{\frac{1}{2}}, f'(k_t) = \frac{1}{2}k_t^{-\frac{1}{2}}.$$

Let

$$n = 0.04, \qquad \delta = \frac{3}{52}, \qquad L_t^f = 1, \qquad \alpha = 0.8, \tau = 0.2, p_{t-1} = 1.$$

Then,

$$\tilde{r} = n + \delta + n\delta = 0.1 = \frac{1}{2}k_t^{-\frac{1}{2}}, \tilde{k} = \widetilde{K}_t = 25, f(\tilde{k}) = 5, w_t = 2.5p_t.$$

From (6) with $B_{t+1} = (1+n)B_t$ and $K_{t+1} = (1+n)K_t$, $r_{t+1} = \tilde{r}$,

$$G_t - \tau w_t L_t + \alpha\left[(1-\tau)w_t L_t + \frac{\tilde{r}-n}{1+\tilde{r}}B_t\right] + p_t n K_t = (1-\tau)w_t L_t + p_t\tilde{r}K_t.$$

Let us consider the steady state in Period t. From (7), $B_t \approx 35.87p_t$. The consumption demand is

$$\alpha\left[(1-\tau)w_t L_t + \frac{\tilde{r}-n}{1+\tilde{r}}B_t\right] \approx 3.17p_t.$$

The invest demand is $p_t n\widetilde{K}_t = 1p_t$. The supply minus tax is

$$(1-\tau)w_t L_t + p_t\tilde{r}K_t = 4.5p_t.$$

If the budget deficit is 0.33, we have $p_t = 1$ and full employment is realized under constant price. If it is larger than 0.33, $p_t > 1$ and an inflation is triggered. On the other hand, with downward price rigidity, if the budget deficit is smaller than 0.33, a recession occurs.

5. Conclusion

In recent years, I have been interested in the issue of budget deficits and government debt, not specifically as a policy to recover from recession, but to prove that budget deficits are necessary to achieve full employment without inflation nor deflation in a steady state. The key to this is to consider a growing economy and the fact that people hold money for liquidity and other reasons. This paper develops the argument not in the overlapping generations model in which people live for a finite period, but in a model in which people live for an infinite period. If similar conclusions can be reached in various models, a model in which people live forever may be easier to handle than a model in which generations overlap.

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Conflict of interest

The author claims that the manuscript is completely original. The author also declares no conflict of interest.

Appendix

A1. Details of calculation in Section 2.

The Lagrange function for maximization of (1) under the budget constraint (3) is written as follows.

$$\begin{aligned}\mathcal{L} = {} & u\left(c_0, \frac{m_1}{p_0}\right) + \frac{1}{1+\delta} u\left(c_1, \frac{m_2}{p_1}\right) + \left(\frac{1}{1+\delta}\right)^2 u\left(c_2, \frac{m_3}{p_2}\right) + \cdots + \left(\frac{1}{1+\delta}\right)^t u\left(c_t, \frac{m_{t+1}}{p_t}\right) + \left(\frac{1}{1+\delta}\right)^{t+1} u\left(c_{t+1}, \frac{m_{t+2}}{p_{t+1}}\right) \\
& + \cdots + \mu_0 \left[(1+r_1)(1-\tau)w_0 l_0 - (1+r_1)p_0 c_0 - r_1 m_1 + \frac{1+r_1}{1+n} b_0 - b_1\right] \\
& + \mu_1 \left[(1+r_2)(1-\tau)w_1 l_1 - (1+r_2)p_1 c_1 - r_2 m_2 + \frac{1+r_2}{1+n} b_1 - b_2\right] \\
& + \mu_2 \left[(1+r_3)(1-\tau)w_2 l_2 - (1+r_3)p_2 c_2 - r_3 m_3 + \frac{1+r_3}{1+n} b_2 - b_3\right] + \cdots \\
& + \mu_t \left[(1+r_{t+1})(1-\tau)w_t l_t - (1+r_{t+1})p_t c_t - r_{t+1} m_{t+1} + \frac{1+r_{t+1}}{1+n} b_t - b_{t+1}\right] \\
& + \mu_{t+1} \left[(1+r_{t+2})(1-\tau)w_{t+1} l_{t+1} - (1+r_{t+2})p_{t+1} c_{t+1} - r_{t+2} m_{t+2} + \frac{1+r_{t+2}}{1+n} b_{t+1} - b_{t+2}\right] + \cdots.\end{aligned}$$

μ_t for $t = 0,1, \cdots, \infty$ are the Lagrange multipliers. The first order conditions for utility maximization over infinite periods are

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{\alpha}{c_t}\left(\frac{1}{1+\delta}\right)^t - \mu_t(1+r_{t+1})p_t = 0, \qquad t = 0,1, \qquad \cdots, \infty,$$

and

$$\frac{\partial \mathcal{L}}{\partial \frac{m_{t+1}}{p_t}} = \frac{(1-\alpha)p_t}{m_{t+1}}\left(\frac{1}{1+\delta}\right)^t - \mu_t r_{t+1} p_t = 0, \qquad t = 0,1, \qquad \cdots, \infty.$$

From them we get

$$p_t c_t = \frac{\alpha}{\mu_t(1+r_{t+1})}\left(\frac{1}{1+\delta}\right)^t, \tag{A.2}$$

and

$$m_{t+1} = \frac{(1-\alpha)}{\mu_t r_{t+1}}\left(\frac{1}{1+\delta}\right)^t.$$

From this

$$\frac{r_{t+1}}{1+r_{t+1}} m_{t+1} = \frac{(1-\alpha)}{\mu_t(1+r_{t+1})}\left(\frac{1}{1+\delta}\right)^t. \tag{A.3}$$

Adding (A.2) and (A.3) yields

$$p_t c_t + \frac{r_{t+1}}{1+r_{t+1}} m_{t+1} = \frac{1}{\mu_t(1+r_{t+1})}\left(\frac{1}{1+\delta}\right)^t. \tag{A.4}$$

From the Lagrange function, we obtain

$$\frac{\partial \mathcal{L}}{\partial b_t} = -\mu_{t-1} + \mu_t \frac{1+r_{t+1}}{1+n} = 0. \tag{A.5}$$

Similarly, for b_{t+1}

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = -\mu_t + \mu_{t+1} \frac{1+r_{t+2}}{1+n} = 0.$$

They are the costate or adjoint equations. I consider its meaning in Section 3.

From (3), (A.2), (A.3) and (A.4), we obtain

$$c_t = \frac{\alpha}{p_t}\left[(1-\tau)w_t l_t + \frac{b_t}{1+n} - \frac{1}{1+r_{t+1}} b_{t+1}\right],$$

and

$$m_{t+1} = (1-\alpha)\frac{1+r_{t+1}}{r_{t+1}}\left[(1-\tau)w_t l_t + \frac{b_t}{1+n} - \frac{1}{1+r_{t+1}} b_{t+1}\right].$$

A2. Proof of Proposition 1

Arranging (6),

$$G_t - \tau w_t L_t + \alpha\left[(1-\tau)w_t L_t + B_t - \frac{1}{1+r_{t+1}}B_{t+1}\right] + p_t(K_{t+1} - K_t) = (1-\tau)w_t L_t + p_t r_t K_t.$$

Further, we get

$$G_t - \tau w_t L_t + \alpha\left[(1-\tau)w_t L_t + B_t - \frac{1}{1+r_{t+1}}B_{t+1}\right] + p_t K_{t+1} = (1-\tau)w_t L_t + p_t(1+r_t)K_t.$$

Then,

$$G_t - \tau w_t L_t + \alpha\left[(1-\tau)w_t L_t + B_t - \frac{1}{1+r_{t+1}}B_{t+1}\right] + S_{t+1} - M_{t+1} = (1-\tau)w_t L_t + \frac{p_t}{p_{t-1}}(B_t - M_t). \qquad (\mathrm{A.6})$$

Note that the savings at the end of Period t is written as

$$S_{t+1} = \frac{\alpha B_{t+1}}{1+r_{t+1}} + (1-\alpha)\left[(1-\tau)w_t L_t + \frac{b_t}{1+n}L_t^f\right] = \frac{\alpha B_{t+1}}{1+r_{t+1}} + (1-\alpha)[(1-\tau)w_t L_t + B_t].$$

Substituting this into (A.6), we obtain

$$G_t - \tau w_t L_t + \alpha\left[(1-\tau)w_t L_t + B_t - \frac{1}{1+r_{t+1}}B_{t+1}\right] + \frac{\alpha B_{t+1}}{1+r_{t+1}} + (1-\alpha)[(1-\tau)w_t L_t + B_t] - M_{t+1}$$
$$= (1-\tau)w_t L_t + \frac{p_t}{p_{t-1}}(B_t - M_t).$$

Rearranging it yields

$$G_t - \tau w_t L_t = \frac{p_t}{p_{t-1}}(B_t - M_t) + M_{t+1} - B_t. \qquad (\mathrm{A.7})$$

Now let us assume that full employment has been achieved until Period $t-1$ without inflation (nor deflation). Then, we have

$$L_{t-1} = L_{t-1}^f,$$

and

$$p_{t-1} = p_t, \qquad c_t = c_{t+1}, \qquad r_t = r_{t+1} = r_{t+2}, \qquad w_t = w_{t+1}, \qquad b_t = b_{t+1}.$$

Also, in the steady state with full employment and constant price, we get

$$B_{t+1} = (1+n)B_t,$$

and

$$M_{t+1} = (1+n)M_t.$$

Then, with $L_t = L_t^f$, that is, full employment in Period t, (A.7) is rewritten as

$$G_t - \tau w_t L_t^f, = M_{t+1} - M_t = nM_t. \qquad (\mathrm{A.8})$$

So long as $0 < \alpha < 1$ and $n > 0$, this is positive. (Q.E.D.)

A3. Proof of Proposition 3

From (A.2) in Appendix 1 for Period t and $t+1$ we have

$$\mu_t = \frac{\alpha}{p_t c_t (1 + r_{t+1})} \left(\frac{1}{1+\delta} \right)^t.$$

Similarly,

$$\mu_{t-1} = \frac{\alpha}{p_{t-1} c_{t-1} (1 + r_t)} \left(\frac{1}{1+\delta} \right)^{t-1}.$$

From the costate equation, (A.5) in Appendix 1,

$$-\mu_{t-1} + \mu_t \frac{1 + r_{t+1}}{1+n} = 0.$$

In the steady state under full employment with constant price, since $p_t = p_{t-1}, c_t = c_{t-1}, r_t = r_{t+1}$,

$$\mu_t = \frac{1}{1+\delta} \mu_{t-1}.$$

Then, we obtain

$$\frac{1 + r_{t+1}}{1+n} = 1 + \delta$$

or

$$1 + r_{t+1} = (1+n)(1+\delta) > 1 + n.$$

Denote the steady state value of the interest rate by

$$\tilde{r} = n + \delta + n\delta.$$

If this relation holds, the capital-labor ratio, k, which is constant in the steady state, satisfies

$$f'(k) = \tilde{r}.$$

Denote this value of k by $\tilde{k}$. Then, the steady state value of the capital in Period t and $t+1$ are

$$\widetilde{K}_t = \tilde{k} L_t^f,$$

and

$$\widetilde{K}_{t+1} = \tilde{k} L_{t+1}^f = (1+n)\widetilde{K}_t.$$

Also we have

$$\widetilde{K}_t = \frac{S_t - M_t}{p_{t-1}} = \frac{B_t - M_t}{(1+\tilde{r})p_{t-1}},$$

and

$$\widetilde{K}_{t+1} = \frac{S_{t+1} - M_{t+1}}{p_t} = \frac{B_{t+1} - M_{t+1}}{(1+\tilde{r})p_t}. \tag{A.9}$$

The steady state value of the nominal wage rate in Period t is

$$w_t = p_t\left[f(\tilde{k}) - f'(\tilde{k})\tilde{k}_t\right].$$

Also in the steady state

$$B_{t+1} = (1+n)B_t. \tag{A.10}$$

Therefore,

$$M_{t+1} = (1-\alpha)\tfrac{1+\tilde{r}}{\tilde{r}}\left[(1-\tau)w_tL_t^f + B_t - \tfrac{1+n}{1+\tilde{r}}B_t\right] = (1-\alpha)\tfrac{1+\tilde{r}}{\tilde{r}}\left[(1-\tau)w_tL_t^f + \tfrac{\tilde{r}-n}{1+\tilde{r}}B_t\right].$$

It is rewritten as

$$M_{t+1} = (1-\alpha)\frac{1+\tilde{r}}{\tilde{r}}(1-\tau)w_tL_t^f + (1-\alpha)\frac{\tilde{r}-n}{\tilde{r}}B_t. \tag{A.11}$$

From (A.9) and (A.10),

$$(1+n)B_t - M_{t+1} = (1+\tilde{r})(1+n)p_t\widetilde{K}_t. \tag{A.12}$$

By (A.11) and (A.12), we obtain

$$B_t = \frac{(1-\alpha)\frac{1+\tilde{r}}{\tilde{r}}(1-\tau)w_tL_t^f + (1+\tilde{r})(1+n)p_t\widetilde{K}_t}{1+n-(1-\alpha)\frac{\tilde{r}-n}{\tilde{r}}}. \tag{A.13}$$

This is the explicit solution of the value of the savings in Period t. By similar calculations, we get

$$\begin{aligned} M_{t+1} &= (1+n)\frac{(1-\alpha)\frac{1+\tilde{r}}{\tilde{r}}(1-\tau)w_tL_t^f + (1+\tilde{r})(1+n)p_t\widetilde{K}_t}{1+n-(1-\alpha)\frac{\tilde{r}-n}{\tilde{r}}} \\ &-(1+\tilde{r})(1+n)p_t\widetilde{K}_t = (1+n)(1-\alpha)\frac{\frac{1+\tilde{r}}{\tilde{r}}(1-\tau)w_tL_t^f + (1+\tilde{r})\frac{\tilde{r}-n}{\tilde{r}}p_t\widetilde{K}_t}{1+n-(1-\alpha)\frac{\tilde{r}-n}{\tilde{r}}}, \end{aligned} \tag{A.14}$$

with

$$\widetilde{K}_t = \tilde{k}L_t^f.$$

This is the explicit solution of the value of the money holding at the end of Period t. By (A.7), (A.13) and (A.14) for M_{t+1} and M_t, we can explicitly get the steady state values of the budget deficit. (Q.E.D.)

A4. Hamiltonian method

Although the Lagrange multiplier method is used in the text, analysis by Hamiltonian method is also possible, which is briefly discussed below. (A.1) is rearranged as follows.

$$\mathcal{L} = u\left(c_0, \frac{m_1}{p_0}\right) + \frac{1}{1+\delta} u\left(c_1, \frac{m_2}{p_1}\right) + \left(\frac{1}{1+\delta}\right)^2 u\left(c_2, \frac{m_3}{p_2}\right) + \cdots + \left(\frac{1}{1+\delta}\right)^t u\left(c_t, \frac{m_{t+1}}{p_t}\right) + \left(\frac{1}{1+\delta}\right)^{t+1} u\left(c_{t+1}, \frac{m_{t+2}}{p_{t+1}}\right)$$
$$+ + \mu_0 \left[(1+r_1)(1-\tau)w_0 l_0 - (1+r_1)p_0 c_0 - r_1 m_1 + \frac{r_1 - n}{1+n} b_0\right] + \mu_0 b_0 - \mu_0 b_1$$
$$+ \mu_1 \left[(1+r_2)(1-\tau)w_1 l_1 - (1+r_2)p_1 c_1 - r_2 m_2 + \frac{r_2 - n}{1+n} b_1\right] + \mu_1 b_1 - \mu_1 b_2$$
$$+ \mu_2 \left[(1+r_3)(1-\tau)w_2 l_2 - (1+r_3)p_2 c_2 - r_3 m_3 + \frac{r_3 - n}{1+n} b_2\right] + \mu_2 b_2 - \mu_2 b_3 + \cdots$$
$$+ \mu_t \left[(1+r_{t+1})(1-\tau)w_t l_t - (1+r_{t+1})p_t c_t - r_{t+1} m_{t+1} + \frac{r_{t+1} - n}{1+n} b_t\right] + \mu_t b_t - \mu_t b_{t+1}$$
$$+ \mu_{t+1} \left[(1+r_{t+2})(1-\tau)w_{t+1} l_{t+1} - (1+r_{t+2})p_{t+1} c_{t+1} - r_{t+2} m_{t+2} + \frac{r_{t+2} - n}{1+n} b_{t+1}\right] + \mu_{t+1} b_{t+1}$$
$$- \mu_{t+1} b_{t+2} + \cdots.$$

Further, we have

$$\mathcal{L} = \mu_0 b_0 + u\left(c_0, \frac{m_1}{p_0}\right) + \mu_0 \left[(1+r_1)(1-\tau)w_0 l_0 - (1+r_1)p_0 c_0 - r_1 m_1 + \frac{r_1 - n}{1+n} b_0\right] + (\mu_1 - \mu_0) b_1$$
$$+ \frac{1}{1+\delta} u\left(c_1, \frac{m_2}{p_1}\right) + \mu_1 \left[(1+r_2)(1-\tau)w_1 l_1 - (1+r_2)p_1 c_1 - r_2 m_2 + \frac{r_2 - n}{1+n} b_1\right] + (\mu_2 - \mu_1) b_2$$
$$+ \left(\frac{1}{1+\delta}\right)^2 u\left(c_2, \frac{m_3}{p_2}\right) + \mu_2 \left[(1+r_3)(1-\tau)w_2 l_2 - (1+r_3)p_2 c_2 - r_3 m_3 + \frac{r_3 - n}{1+n} b_2\right] + \cdots + (\mu_t$$
$$- \mu_{t-1}) b_t + \left(\frac{1}{1+\delta}\right)^t u\left(c_t, \frac{m_{t+1}}{p_t}\right)$$
$$+ \mu_t [(1+r_{t+1})(1-\tau)w_t l_t - (1+r_{t+1})p_t c_t - r_{t+1} m_{t+1} + \frac{r_{t+1} - n}{1+n} b_t\Big] + (\mu_{t+1} - \mu_t) b_{t+1}$$
$$+ \left(\frac{1}{1+\delta}\right)^{t+1} u\left(c_{t+1}, \frac{m_{t+2}}{p_{t+1}}\right)$$
$$+ \mu_{t+1} [(1+r_{t+2})(1-\tau)w_{t+1} l_{t+1} - (1+r_{t+2})p_{t+1} c_{t+1} - r_{t+2} m_{t+2} + \frac{r_{t+2} - n}{1+n} b_{t+1}\Big] + \cdots.$$

Let define the present value Hamiltonian by

$$H_t = \left(\frac{1}{1+\delta}\right)^t u\left(c_t, \frac{m_{t+1}}{p_t}\right) + \mu_t \left[(1+r_{t+1})(1-\tau)w_t l_t - (1+r_{t+1})p_t c_t - r_{t+1} m_{t+1} + \frac{r_{t+1} - n}{1+n} b_t\right]$$

with

$$u\left(c_t, \frac{m_{t+1}}{p_t}\right) = \alpha \ln c_t + (1-\alpha) \ln \frac{m_{t+1}}{p_t}, \qquad 0 < \alpha < 1.$$

The first order conditions for life time utility maximization are

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{\partial H_t}{\partial c_t} = \frac{\alpha}{c_t}\left(\frac{1}{1+\delta}\right)^t - \mu(1+r_{t+1})p_t = 0,$$

and

$$\frac{\partial \mathcal{L}}{\partial \frac{m_{t+1}}{p_t}} = \frac{\partial H_t}{\partial \frac{m_{t+1}}{p_t}} = \frac{(1-\alpha)p_t}{m_{t+1}}\left(\frac{1}{1+\delta}\right)^t - \mu_t r_{t+1} p_t = 0.$$

From them we obtain

$$p_t c_t = \frac{\alpha}{\mu_t(1+r_{t+1})}\left(\frac{1}{1+\delta}\right)^t,$$

and

$$\frac{r_{t+1}}{1+r_{t+1}} m_{t+1} = \frac{1-\alpha}{\mu_t(1+r_{t+1})}\left(\frac{1}{1+\delta}\right)^t.$$

They are equivalent to (A.2) and (A.3). The costate equation is

$$\mu_t - \mu_{t-1} + \frac{\partial H_t}{\partial b_t} = \mu_t - \mu_{t-1} + \mu_t \frac{r_{t+1} - n}{1+n} = -\mu_{t-1} + \mu_t \frac{1+r_{t+1}}{1+n} = 0.$$

This is equivalent to (A.5).

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