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Budget deficit and money holding when consumers live forever in an endogenous growth model

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ABSTRACT

In this paper I will show that budget deficit (or fiscal deficit) is necessary to achieve full employment under constant prices or inflation, using a model of endogenous growth in which consumers hold money for the reason of liquidity and live forever. Budget deficit need not be offset by future budget surpluses. I consider the continuous time case by taking the limit of the discrete time case when the time interval approaches zero. A continuous time dynamic model seems to be more general than a discrete time model. When the actual budget deficit is greater (smaller) than the value which is necessary and sufficient for full employment under constant prices, an inflation (a recession) occurs. The main argument of this paper is that a growing economy requires the continuation of budget deficit, and that we should not think of paying off the resulting government debt with taxes.

KEYWORDS

Budget deficit; endogenous growth model; infinitely living consumers; continuous time model; functional finance

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1. Introduction

In this paper I will show that budget deficit (or fiscal deficit) is necessary to achieve full employment under constant price or inflation, using a model of endogenous growth in which consumers hold money for the reason of liquidity and live forever. When the actual budget deficit is greater (smaller) than the value which is necessary and sufficient for full employment under constant price, an inflation (a recession) occurs. The main argument of this paper is that a growing economy requires the continuation of budget deficit, and that we should not think of paying off the resulting government debt with taxes.

In recent some studies I have examined the budget deficit in a growing economy when consumers get utility from money holding as well as consumptions of goods. In these studies I have used several types of model for consumers' behavior and economic growth. About consumers' behavior one is a two-periods overlapping generations model in which consumers live over two periods, the younger period and the older period. They work in the younger period and retire in the older period. They consume goods in the older period by the savings carried over from the younger period. About the overlapping generations model I referred to Diamond (1965), J. Tanaka (2010, 2011a, 2011b, 2013) and Otaki (2007, 2009, 2015). Using an overlapping generations model, I have shown the necessity or inevitability of budget deficit in an growing economy. The main difference from the work of others is that I assume that consumers derive utility not only from the consumption of goods but also from the holding of money for reasons such as securing liquidity.

Such a model may be reasonably realistic, but it may not be very general. Therefore, I would like to examine the same issue under the assumption that people will live forever. One such model is a discrete time dynamic model. A reference for the discrete time dynamic optimization is Tachibana (2006). It may be considered sufficiently general than the overlapping generations model, but it still seems to lack generality. In this study I examine the problem by a general continuous time dynamic model. About a continuous time model in which consumers infinitely live I refer to Weil (1987, 1989). However, in this paper I consider the continuous time case by taking the limit of the discrete time case when the time interval approaches zero.

As for economic growth, I have mainly used a model of exogenous growth by population growth or technological progress. In this paper I use an endogenous growth model in which the economy grows by investment of firms. About an endogenous growth model I refer to Grossman and Yanagawa (1993) and, mainly, Maebayashi and J. Tanaka (2021), but our model is a simplified version adding money holding of consumers. In Y. Tanaka (2024) I have used an overlapping generations version of Krugman's world's smallest macroeconomic model in which a good is produced only by labor.

The model of this paper itself is neoclassical, but its spirit may be post-Keynesian in that it does not abandon the goal of full employment out of a dislike of budget deficit or the accumulation of government debt. Lopez-Gallardo(2000) is a study of budget deficit and full employment from a post-Keynesian standpoint. It is inspired by Minsky (1986) and related to Mostler (1997-1998), Wray (1998) and Kregel (1998). Mostler, Wray and Kregel (M-W-K), as a policy for full employment, proposed the following:

"Let the government assume the role of employer of last resort at a given wage rate, so that anybody willing to work at that rate will get a job from the government. Government expenditure will thus expand, but will not entail any complication because "the purchasing ability of the government is limited only by what is available for sale in exchange for dollars" (Mosler, 1997-1998, p.169), while this availability, we are told, is elastic below full employment. Now, government expenditure will grow probably over and above tax receipts, and a budget deficit will ensue. However, M-W-K demonstrate, with explanations rich in theoretical, historical, and institutional details, that the government can simply create enough new money, or otherwise sell securities, to finance the deficit with an unchanging rate of interest." (Lopez-Gallardo (2000), p. 550).

Lopez-Gallardo(2000) discusses various problems with this proposal, which are not of interest to this paper.

In Japan and many other countries, budget deficit and accumulated government debt have become a problem, and it is argued that fiscal soundness must be improved. Is this really the case? Even if the goal of achieving a balanced budget will worsen the economy, impede growth, and create unemployment, does it still make sense to improve fiscal soundness? These questions are the starting point for this study.

The significance of government debt and budget deficit, and intergenerational burden have been analyzed by J. Tanaka (Jumpei Tanaka, references above mentioned). He focuses on the intergenerational economic welfare gap due to the presence or absence of government debt, but his main model does not include economic growth and assumes that all government debt is redeemed by taxes. The interest of this paper lies elsewhere. I am interested in proving that we need budget deficit in a growing economy where consumers hold money.

In this paper I mainly consider budget deficit due to money issuance, not government bonds. About the relation of government bonds and money issuance please see Oguri (2011) and Appendix 2 of this paper.

Lerner's famous functional finance theory (Lerner (1943), (1944)) does not consider whether the government should run surplus or deficit to be meaningful in and of itself, but believes that fiscal policy should be used to achieve near full employment while avoiding inflation as much as possible. Lerner(1943) said:

"The central idea is that government fiscal policy, its spending and taxing, its borrowing and repayment of loans, its issue of new money and its withdrawal of money, shall all be undertaken with an eye only to the results of these actions on the economy and not to any established traditional doctrine about what is sound or unsound. This principle of judging only by effects has been applied in many other fields of human activity, where it is known as the method of science as opposed to scholasticism. The principle of judging fiscal measures by the way they work or function in the economy we may call Functional Finance. The first financial responsibility of the government (since nobody else can undertake that responsibility) is to keep the total rate of spending in the country on goods and services neither greater nor less than that rate which at the current prices would buy all the goods that it is possible to produce. If total spending is allowed to go above this there will be inflation, and if it is allowed to go below this there will be unemployment. The government can increase total spending by spending more itself or by reducing taxes so that the taxpayers have more money left to spend. It can reduce total spending by spending less itself or by raising taxes so that taxpayers have less money left to spend. By these means total spending can be kept at the required level, where it will be enough to buy the goods that can be produced by all who want to work, and yet not enough to bring inflation by demanding (at current prices) more than can be produced".

In this paper, I follow Lerner's Functional Finance. For more on Functional Finance, see Forstater (1999).

In the next section, I present the model of this paper and prove that a budget deficit is necessary to achieve full employment under constant goods prices. I also show that if the actual budget deficit is greater than what is necessary and sufficient for full employment under constant prices, an inflation occurs; if the actual budget deficit is less than what is necessary and sufficient for full employment, a recession occurs. Therefore, balanced budget cannot achieve full employment under constant prices or inflation. I do not assume that budget deficit must later be made up by budget surplus. I will analyze by the Lagrangian method in the next section. I present also a solution by Hamiltonian methods in Appendix 3. In Section 3 I calculate the explicit values of the savings and money holding in the steady state. In Section 4 I present some empirical data about budget deficit and government debt. Section 5 is a concluding section. In Appendix 1 I will show that if money as well as goods is produced by capital and labor, and the value of money equals its production cost, then budget deficit is not necessary for full employment under constant prices. However, if money circulates with a value greater than the cost of production, a budget deficit will be necessary. In that case the difference between the value of money and its production cost is the so-called *seigniorage*. A moderate seigniorage is necessary for economic growth without inflation unless the production of money is quite costly. In Appendix 2 I consider the case where financial assets are held not in money but in interest-producing government bonds which have almost the same liquidity as money. I will show that the debt-GDP ratio should be constant, and as the larger the propensity to consume is, the smaller the debt-GDP ratio is. As Blanchard (2022) notes, many discussions of debt-GDP ratio use simple calculations based on comparisons of primary budget balances, the interest rate, and the growth rate. But is the argument not so simple? Assuming a steady state of full employment, which may or may not include inflation, the size of the budget deficit to achieve this is naturally determined, and the larger the budget deficit is, the higher the inflation rate is. On the other hand, the larger (smaller) the propensity to consume is, the smaller (larger) the budget deficit required to achieve full employment under a constant rate of price increase is. Therefore, the larger the propensity to consume is, the less likely it is that the debt-GDP ratio will become large.

This paper is an example of an analysis, using a very simple model, of the following statement by J. M. Keynes.

"Unemployment develops, that is to say, because people want the moon; - men cannot be employed when the object of desire (i.e. money) is something which cannot be produced and the demand for which cannot be readily choked off. There is no remedy but to persuade the public that green cheese is practically the same thing and to have a green cheese factory (i.e. a central bank) under public control." (Keynes(1936), Chap. 17)

The following discussion mainly focuses on the steady state without inflation, but this does not mean assuming price rigidity, but rather examining the government's fiscal policy to achieve and maintain full employment while preventing inflation from occurring. Depending on the size of the budget deficit, full employment with inflation may be considered[1].

2. Money holding and budget deficit in a growing economy

2.1. Consumers' behavior

I consider an endogenous growth model in which consumers infinitely live and hold money for the reason of liquidity and so on.

First, I consider a discrete time model of consumers' behavior, and from the discrete time version I derive a continuous time model. A model assuming discrete time would be easier to deal with, but time division is arbitrary and a model of continuous time would be more general.

Denote the unit interval of period by

$$\Delta t = \frac{1}{T}$$

T is a positive integer. Let s be a nonnegative integer, and denote

$$t = s\Delta t$$

Let c_t be the real value of the consumption by a consumer, and p_t be the price of the good in Period t. Let $m_{t+\Delta t}$ be the nominal value of the money holding of the consumer at the end of Period t. It is carried over to the next period, Period $t+\Delta t$. Therefore, $\frac{m_{t+\Delta t}}{p_t}$ is the real value of the money holding at the end of Period t.

The consumers derive utility from the consumption and the real value of the money holding. The consumer's utility over an infinite period of time is expressed as follows;

[1] The calculation for the expected inflation case can be obtained by replacing the (real) growth rate by a nominal growth rate that includes inflation.

$$\sum_{s=0}^{\infty}\left(\frac{1}{1+\delta\Delta t}\right)^{\frac{t}{\Delta t}}u\left(c_t,\frac{m_{t+\Delta t}}{p_t}\right)\Delta t \tag{1}$$

$\delta > 0$ is the discount rate. Specifically, the utility function is

$$u\left(c_t,\frac{m_{t+\Delta t}}{p_t}\right)=\alpha lnc_t+(1-\alpha)ln\frac{m_{t+\Delta t}}{p_t}$$

α is the parameter of the utility function, and $0<\alpha<1$.

Let $s_{t+\Delta t}$ be the savings of the consumer at the end of Period t, and $b_{t+\Delta t}$ be its value in Period $t+\Delta t$. Then, we have

$$b_{t+\Delta t}=(1+r_{t+\Delta t}\Delta t)(s_{t+\Delta t}-m_{t+\Delta t})+m_{t+\Delta t} \tag{2}$$

$s_{t+\Delta t}-m_{t+\Delta t}$ represents the portion of the savings that is invested in productive capital, which generates interest in the next period. $r_{t+\Delta t}$ is the interest rate (or the rate of return) of the capital in Period $t+\Delta t$.

Similarly,

$$b_t=(1+r_t\Delta t)(s_t-m_t)+m_t$$

From (2),

$$s_{t+\Delta t}=\frac{b_{t+\Delta t}-m_{t+\Delta t}}{1+r_{t+\Delta t}\Delta t}+m_{t+\Delta t}=\frac{b_{t+\Delta t}}{1+r_{t+\Delta t}\Delta t}+\frac{r_{t+\Delta t}\Delta t}{1+r_{t+\Delta t}\Delta t}m_{t+\Delta t}$$

Let w_t be the nominal wage rate, and l_t be the indicator that represents whether the consumer is employed or not. Let τ be the tax rate, $0<\tau<1$. Then, the budget constraint for the consumer in period t is

$$p_tc_t\Delta t+s_{t+\Delta t}=(1-\tau)w_tl_t\Delta t+b_t,\ \ t=0,1,\cdots,\infty$$

If full employment is achieved, $l_t=1$ for all consumers. I assume that the savings at the end of Period t are distributed equally among consumers in Period $t+\Delta t$. The budget constraint is rewritten as follows;

$$p_tc_t\Delta t+\frac{b_{t+\Delta t}}{1+r_{t+\Delta t}\Delta t}+\frac{r_{t+\Delta t}\Delta t}{1+r_{t+\Delta t}\Delta t}m_{t+\Delta t}=(1-\tau)w_tl_t\Delta t+b_t,\ \ t=0,1,\cdots,\infty$$

From this,

$$p_tc_t\Delta t+\frac{r_{t+\Delta t}\Delta t}{1+r_{t+\Delta t}\Delta t}m_{t+\Delta t}=(1-\tau)w_tl_t\Delta t+b_t-\frac{b_{t+\Delta t}}{1+r_{t+\Delta t}\Delta t},t=0,1,\cdots,\infty$$

It is rewritten as

$$\begin{gathered}b_{t+\Delta t}=(1+r_{t+\Delta t}\Delta t)(1-\tau)w_tl_t\Delta t-(1+r_{t+\Delta t}\Delta t)p_tc_t\Delta t-r_{t+\Delta t}\Delta tm_{t+\Delta t}\\+(1+r_{t+\Delta t}\Delta t)b_t,\ \ t=0,1,\cdots,\infty\end{gathered} \tag{3}$$

Further, we get

$$b_{t+\Delta t}-b_t=(1+r_{t+\Delta t}\Delta t)(1-\tau)w_tl_t\Delta t-(1+r_{t+\Delta t}\Delta t)p_tc_t\Delta t$$

$$-r_{t+\Delta t}\Delta tm_{t+\Delta t}+r_{t+\Delta t}\Delta tb_t\ \ t=0,1,\cdots,\infty$$

From this, we obtain

$$\frac{b_{t+\Delta t}-b_t}{\Delta t}=(1+r_{t+\Delta t}\Delta t)(1-\tau)w_t l_t-(1+r_{t+\Delta t}\Delta t)p_t c_t-r_{t+\Delta t}m_{t+\Delta t}$$

$$+r_{t+\Delta t}b_t, t=0,1,\cdots,\infty$$

Now suppose

$$\Delta t\to 0$$

Then, we get

$$\frac{\partial b_t}{\partial t}=(1-\tau)w_t l_t-p_t c_t-r_t m_t+r_t b_t \tag{4}$$

Let $g>0$ be the real growth rate. It is determined by investment of the firms. I will analyze it in the next subsection. In the steady state we can assume

$$b_{t+\Delta t}=(1+g\Delta t)b_t$$

Then,

$$\frac{b_{t+\Delta t}-b_t}{\Delta t}=\frac{g\Delta t b_t}{\Delta t}=gb_t$$

This means

$$\frac{\partial b_t}{\partial t}=gb_t$$

Therefore, we obtain

$$(1-\tau)w_t l_t-p_t c_t-r_t m_t+(r_t-g)b_t=0 \tag{5}$$

Also in the steady state for c_t and m_t we have

$$\frac{\partial c_t}{\partial t}=gc_t,\frac{\partial m_t}{\partial t}=gm_t$$

and so on. Let

$$\eta=\frac{1}{\delta\Delta t}$$

then

$$\left(\frac{1}{1+\delta\Delta t}\right)^{\frac{t}{\Delta t}}=(1+\delta\Delta t)^{-\frac{t}{\Delta t}}=\left(1+\frac{1}{\eta}\right)^{-\eta\delta t}$$

Considering the limit of this equation when $\Delta t\to 0$,

$$\lim_{\Delta t\to 0}\left(\frac{1}{1+\delta\Delta t}\right)^{\frac{t}{\Delta t}}=\lim_{\eta\to\infty}\left(1+\frac{1}{\eta}\right)^{-\eta\delta t}=e^{-\delta t}$$

Also, when $\Delta t \rightarrow 0$, we obtain

$$\sum_{s=0}^{\infty}\left(\frac{1}{1+\delta\Delta t}\right)^{\frac{t}{\Delta t}}u\left(c_t,\frac{m_{t+\Delta t}}{p_t}\right)\Delta t\int_{t=0}^{\infty}e^{-\delta t}u\left(c_t,\frac{m_t}{p_t}\right)dt$$

This is the continuous time version of (1). By the utility maximization in the steady state we obtain

$$c_t=\frac{\alpha}{p_t}[(1-\tau)w_t l_t+(r_t-g)b_t] \tag{6}$$

and

$$m_t=\frac{1-\alpha}{r_t}[(1-\tau)w_t l_t+(r_t-g)b_t] \tag{7}$$

In addition we can show that the equilibrium interest rate is

$$r_t=g+\delta \tag{8}$$

About details of calculation please see Appendix 3. In Appendix 3 we also present analysis by Hamiltonian method.

Denote the labor supply or the employment under full employment by L^f. It is constant. Also, we denote

$$B_t=b_tL^f, S_t=s_tL^f, C_t=c_tL^f, M_t=m_tL^f$$

In the continuous time case $B_t=S_t$. They are the total values of the per capita values b_t, s_t, c_t and m_t under full employment. The real value of the capital is

$$K_t=\frac{B_t-M_t}{p_t}$$

The capital per labor under full employment is

$$\frac{K_t}{L^f}$$

2.2. Firms' behavior

Let Y_t, K_t and L_t be the output, capital input and labor input. The number of firms is normalized to one, and the labor supply is also normalized to one, that is, $L^f=1$ (and $0<L_t\leqq 1$). In Period t the production function of each firm is

$$Y_t=K_t^{\beta}\left(\theta\widetilde{K}_tL_t\right)^{1-\beta}, 0<\beta<1$$

θ is a positive constant. It is larger than 1. $\theta\widetilde{K}_t$ is the labor-augmenting productivity. $\widetilde{K}_t$ is the average capital per population, that is, $\widetilde{K}_t=\frac{K_t}{L^f}$. However, K_t in this formulation is the capital over all the economy, and $\widetilde{K}_t$ is given for the firms. The profit of a firm is

$$\pi_t=p_tK_t^{\beta}\left(\theta\widetilde{K}_tL_t\right)^{1-\beta}-r_tp_{t-\Delta t}K_t-w_tL_t$$

$p_{t-\Delta t}K_t$ is the nominal amount of the capital at the time the investment is made, Period $t-\Delta t$. K_t is the amount of capital in Period t, but the investment in it takes place one period earlier, in Period $t-\Delta t$, so $p_{t-\Delta t}$, not p_t, is the price of that capital. The first order conditions for profit maximization are

$$p_t\beta K_t^{\beta-1}\left(\theta\widetilde{K}_tL_t\right)^{1-\beta}=r_tp_{t-\Delta t}$$

and

$$p_t\theta\widetilde{K}_t(1-\beta)K_t^{\beta}\left(\theta\widetilde{K}_tL_t\right)^{-\beta}=w_t$$

In the equilibrium $\widetilde{K}_t=K_t$. Thus,

$$r_tp_{t-\Delta t}=p_t\beta K_t^{\beta-1}(\theta K_tL_t)^{1-\beta}=p_t\beta(\theta L_t)^{1-\beta}$$

or

$$r_t=\frac{p_t}{p_{t-\Delta t}}\beta(\theta L_t)^{1-\beta}$$

and

$$w_t=p_t\theta K_t(1-\beta)K_t^{\beta}(\theta K_tL_t)^{-\beta}=p_t(1-\beta)\theta^{1-\beta}L_t^{-\beta}K_t$$

The wage rate is proportional to the capital input. From them, we obtain

$$p_{t-\Delta t}r_tK_t=p_t\beta(\theta L_t)^{1-\beta}K_t$$

$$w_tL_t=p_t(1-\beta)(\theta L_t)^{1-\beta}K_t$$

$$p_{t-\Delta t}r_tK_t+w_tL_t=p_t(\theta L_t)^{1-\beta}K_t$$

and

$$p_tY_t=p_tK_t^{\beta}(\theta K_tL_t)^{1-\beta}=p_t(\theta L_t)^{1-\beta}K_t=p_{t-\Delta t}r_tK_t+w_tL_t$$

Let $\Delta t\to 0$, then under full employment,

$$r_t=\beta\theta^{1-\beta},w_t=p_t(1-\beta)\theta^{1-\beta}K_t$$

2.3. Market equilibrium

We consider the steady state with constant prices, constant growth rate and full employment. Let

$$1+g\Delta t=\frac{K_t}{K_{t-\Delta t}}$$

Then,

$$K_t-K_{t-\Delta t}=g\Delta tK_{t-\Delta t}$$

From this, we have

$$\frac{\partial K_t}{\partial t}=gK_t$$

It is an increase of the capital, that is, the investment. The total nominal consumption demand in the steady state under full employment with constant prices is

$$p_t C_t = L^f p_t c_t = \alpha[(1-\tau)w_t + (r_t - g)b_t]L^f = \alpha\left[(1-\tau)w_t L^f + (r_t - g)B_t\right]$$

The total money holding is

$$M_t = L^f m_t = \frac{1-\alpha}{r_t}[(1-\tau)w_t + (r_t - g)b_t]L^f = \frac{1-\alpha}{r_t}\left[(1-\tau)w_t L^f + (r_t - g)B_t\right] \tag{9}$$

Let G_t be the fiscal expenditure in Period t. The total nominal demand is

$$G_t + \alpha\left[(1-\tau)w_t L^f + (r_t - g)B_t\right] + p_t g K_t$$

The market clearing condition is

$$G_t + \alpha\left[(1-\tau)w_t L^f + (r_t - g)B_t\right] + p_t g K_t = w_t L^f + p_t r_t K_t$$

Arranging it, we have

$$G_t - \tau w_t L^f + \alpha\left[(1-\tau)w_t L^f + (r_t - g)B_t\right] + p_t g K_t = (1-\tau)w_t L^f + p_t r_t K_t$$

Further, since $p_t K_t = B_t - M_t$, we get

$$G_t - \tau w_t L^f + \alpha\left[(1-\tau)w_t L^f + (r_t - g)B_t\right] + g(B_t - M_t) = (1-\tau)w_t L^f + r_t(B_t - M_t)$$

Then,

$$G_t - \tau w_t L^f + \alpha\left[(1-\tau)w_t L^f + (r_t - g)B_t\right] + (g - r_t)B_t - gM_t + r_t M_t = (1-\tau)w_t L^f \tag{10}$$

From (9),

$$r_t M_t = (1-\alpha)\left[(1-\tau)w_t L^f + (r_t - g)B_t\right]$$

Substituting this into (11), we get

$$G_t - \tau w_t L^f + \left[(1-\tau)w_t L^f + (r_t - g)B_t\right] + (g - r_t)B_t - gM_t = (1-\tau)w_t L^f$$

Thus, we obtain

$$G_t - \tau w_t L^f = gM_t \tag{11}$$

So long as $0 < \alpha < 1$ and $g > 0$, this is positive. (10) means that the budget deficit is equal to the increase in the money holding. We have shown the following result.

Proposition 1

In a growing economy, if consumers get utility from money holding, in the steady state under full employment with constant prices, we need positive budget deficit.

Taking M_t as given, if full employment has been achieved prior to that time, then it is necessary and sufficient to create the budget deficit shown in (11) to continue inflation-free full employment in Period t.

Inflation and recession

If the actual budget deficit is larger than the value in (11), M_t should be larger. This will be realized through an increase in income. However, under full employment the real income can not increase. Therefore, only the nominal income increases with constant real income. Then, p_t should be larger, and inflation is triggered. On the other hand, if the actual budget deficit is smaller than the value in (11), M_t should be smaller. This will be realized through a decline in income and production. We have shown the following results.

Proposition 2

If the budget deficit is larger than the value which is necessary and sufficient to achieve full employment under constant prices, inflation is triggered. On the other hand, if the budget deficit is smaller than the value which is necessary and sufficient to achieve full employment under constant prices, a recession occurs.

3. Explicit values of the savings and money holding

Denote the equilibrium interest rate by $\tilde{r}$. Then,

$$\tilde{r} = \beta\theta^{1-\beta}$$

From (8) in the steady state,

$$\tilde{r} = g + \delta$$

or

$$g = \tilde{r} - \delta$$

This is the equilibrium value of the real growth rate. The equilibrium capital-labor ratio, k_t, which is constant in the steady state under constant population, satisfies

$$\frac{\partial k_t}{\partial t} = gk_t$$

Denote this value of k_t by $\tilde{k}$. Then, the steady state value of the capital in Period t is[2]

$$\widetilde{K}_t = \tilde{k}L^f$$

Also we have

$$\widetilde{K}_t = \frac{B_t - M_t}{p_t} \tag{12}$$

The steady state value of the nominal wage rate in Period t is

$$w_t = p_t(1-\alpha)\theta^{1-\beta}\widetilde{K}_t$$

The steady state value of the money holding is

$$M_t = (1-\alpha)\frac{1}{\tilde{r}}(1-\tau)w_tL^f + (1-\alpha)\frac{\tilde{r}-g}{\tilde{r}}B_t \tag{13}$$

From (12),

$$B_t - M_t = p_t\widetilde{K}_t \tag{14}$$

By (13) and (14), we obtain

[2] Since $L^f = 1$ is assumed, k_t and K_t are mutually interchangeable.

$$B_t = \frac{(1-\alpha)\frac{1}{\tilde{r}}(1-\tau)w_t L^f + p_t \widetilde{K}_t}{1-(1-\alpha)\frac{\tilde{r}-g}{\tilde{r}}} \quad (15)$$

This is the explicit solution of the value of the savings in Period t. Then, we get

$$\begin{aligned} M_t &= \frac{(1-\alpha)\frac{1}{\tilde{r}}(1-\tau)w_t L^f + p_t \widetilde{K}_t}{1-(1-\alpha)\frac{\tilde{r}-g}{\tilde{r}}} - p_t \widetilde{K}_t = \frac{(1-\alpha)\frac{1}{\tilde{r}}(1-\tau)w_t L^f + (1-\alpha)\frac{\tilde{r}-g}{\tilde{r}} p_t \widetilde{K}_t}{1-(1-\alpha)\frac{\tilde{r}-g}{\tilde{r}}} \\ &= (1-\alpha)\frac{(1-\tau)w_t L^f + (\tilde{r}-g)p_t \widetilde{K}_t}{\tilde{r}-(1-\alpha)(\tilde{r}-g)} = (1-\alpha)\frac{(1-\tau)w_t L^f + (\tilde{r}-g)p_t \widetilde{K}_t}{\alpha\tilde{r}+(1-\alpha)g} \end{aligned} \quad (16)$$

This is the explicit solution of the value of the money holding at the end of Period t. By (11) and (16), we can explicitly get the steady state value of the budget deficit.

4. Some data

Table 1 shows government deficits in recent years for several representative nations, and Table 2 shows government debt (from OECD Economic Outlook, 2023).

Table 1. Government deficit (% of GDP).

	Average	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
Australia	-3.57	-4.60	-2.80	-2.80	-2.20	-2.20	-1.60	-1.20	-1.40	-7.00	-9.20	-4.30
Austria	-2.35	-2.60	-2.20	-2.00	-2.70	-1.00	-1.50	-0.80	0.20	0.60	-8.00	-5.80
Belgium	-3.43	-4.30	-4.30	-3.10	-3.10	-2.40	-2.40	-0.70	-0.90	-2.00	-9.00	-5.50
Canada	-2.06	-3.30	-2.50	-1.50	0.20	-0.10	-0.50	-0.10	0.40	0.00	-10.90	-4.40
Czech Republic	-1.65	-2.70	-3.90	-1.30	-2.10	-0.60	0.70	1.50	0.90	0.30	-5.80	-5.10
Denmark	0.37	-2.10	-3.50	-1.20	1.10	-1.30	-0.10	1.80	0.80	4.10	0.40	4.10
Finland	-2.17	-1.00	-2.20	-2.50	-3.00	-2.40	-1.70	-0.70	-0.90	-0.90	-5.60	-3.00
France	-4.48	-5.20	-5.00	-4.10	-3.90	-3.60	-3.60	-3.00	-2.30	-3.10	-9.00	-6.50
Germany	-0.13	-0.90	0.00	0.00	0.60	1.00	1.20	1.30	1.90	1.50	-4.30	-3.70
Greece	-5.16	-10.50	-9.10	-13.40	-3.70	-5.90	0.20	0.60	0.90	0.90	-9.70	-7.10
Hungary	-3.45	-5.20	-2.30	-2.60	-2.80	-2.00	-1.80	-2.50	-2.10	-2.00	-7.50	-7.10
Ireland	-3.75	-13.60	-8.50	-6.40	-3.60	-2.00	-0.80	-0.30	0.10	0.50	-5.00	-1.60
Italy	-3.84	-3.60	-2.90	-2.90	-3.00	-2.60	-2.40	-2.40	-2.20	-1.50	-9.70	-9.00
Japan	-5.60	-9.00	-8.20	-7.60	-5.60	-3.70	-3.60	-3.10	-2.50	-3.00	-9.10	-6.20
Korea	1.01	1.00	1.00	1.30	1.20	1.20	2.20	2.70	3.00	1.00	-2.70	-0.80
Netherlands	-1.52	-4.40	-3.90	-3.00	-2.30	-1.90	0.10	1.40	1.50	1.80	-3.70	-2.30
New Zealand	-1.57	-4.30	-2.40	-0.70	0.20	0.00	1.00	1.50	0.80	-0.70	-8.00	-4.70
Norway	7.59	13.30	13.70	10.60	8.60	6.00	4.00	5.00	7.80	6.50	-2.60	10.60
Poland	-2.99	-5.00	-3.80	-4.30	-3.70	-2.60	-2.40	-1.50	-0.20	-0.70	-6.90	-1.80
Portugal	-4.05	-7.70	-6.20	-5.10	-7.40	-4.40	-1.90	-3.00	-0.30	0.10	-5.80	-2.90
Slovak Republic	-3.09	-4.30	-4.40	-2.90	-3.10	-2.70	-2.60	-1.00	-1.00	-1.20	-5.40	-5.40
Slovenia	-4.21	-6.60	-4.00	-14.60	-5.50	-2.80	-1.90	-0.10	0.70	0.70	-7.60	-4.60
Spain	-6.39	-9.70	-11.60	-7.50	-6.10	-5.30	-4.30	-3.10	-2.60	-3.10	-10.10	-6.90
Sweden	-0.31	-0.30	-1.10	-1.50	-1.50	0.00	1.00	1.40	0.80	0.60	-2.80	0.00
Switzerland	0.10	0.70	0.20	-0.40	-0.20	0.50	0.20	1.10	1.30	1.30	-3.10	-0.50
United Kingdom	-5.70	-7.40	-8.00	-5.40	-5.60	-4.60	-3.30	-2.50	-2.30	-2.50	-13.10	-8.00
United States	-7.76	-11.00	-9.20	-5.80	-5.20	-4.60	-5.40	-4.40	-6.10	-6.70	-14.90	-12.10
Average	-2.60	-4.23	-3.60	-3.36	-2.39	-1.85	-1.16	-0.45	-0.14	-0.54	-7.00	-3.87

Table 2. Government debt (% of GDP).

	Average	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
Australia	67.3	45.8	58.6	55.1	61.1	64.4	69.0	66.3	66.9	77.0	92.1	84.4
Austria	97.6	91.5	97.3	94.4	101.9	101.3	102.5	96.3	90.9	89.3	107.1	101.1
Belgium	123.9	111.6	121.0	118.6	130.6	126.2	127.7	120.8	117.8	119.6	139.8	129.1
Canada	116.8	111.1	113.7	107.8	108.6	114.4	115.4	111.8	109.8	111.9	146.1	134.1
Czech Republic	48.2	47.0	56.2	56.0	54.8	51.7	47.4	43.3	40.1	37.8	47.0	48.5
Denmark	54.0	60.1	60.6	56.7	59.1	53.4	51.5	49.0	47.2	48.4	58.4	49.4
Finland	75.6	60.9	68.0	68.8	75.9	79.9	80.5	77.9	75.0	75.2	87.2	82.2
France	122.1	103.8	111.9	112.5	120.2	120.8	123.7	122.9	120.7	123.1	145.9	138.1
Germany	78.6	86.3	88.7	84.1	83.9	79.8	77.0	72.4	69.2	67.6	78.4	77.1
Greece	188.4	113.6	167.1	181.4	183.1	184.2	188.9	193.0	199.3	200.8	236.8	224.3
Hungary	94.4	95.1	98.4	97.2	100.6	98.8	98.7	93.2	86.9	83.9	97.1	88.6
Ireland	92.9	110.8	129.4	132.3	121.2	88.3	85.3	76.2	74.3	68.9	71.3	64.1
Italy	152.0	117.2	135.4	143.2	155.6	156.9	154.6	152.0	146.9	154.2	183.1	172.5
Japan	235.1	218.0	226.6	229.7	234.4	233.3	231.4	230.3	234.2	234.8	257.0	256.0
Korea	51.5	45.3	47.5	47.9	50.7	52.5	51.6	49.4	50.4	52.7	58.9	59.6
Netherlands	73.4	73.6	79.4	78.8	83.3	79.6	77.6	70.8	66.0	62.3	70.2	65.7
New Zealand	42.4	42.4	45.6	42.4	42.2	41.8	40.8	39.5	37.4	34.1	46.9	53.3
Norway	42.0	34.5	35.6	36.2	34.5	40.0	44.0	44.3	45.1	46.4	52.9	48.9
Poland	68.7	62.6	66.0	67.1	71.9	70.3	73.4	69.0	66.7	63.6	77.5	68.1
Portugal	140.8	109.9	137.3	141.1	150.7	148.4	144.3	143.1	137.2	135.6	157.1	143.7
Slovak Republic	66.0	51.1	60.7	65.1	67.8	66.2	67.6	65.3	63.3	62.9	77.6	78.4
Slovenia	85.0	51.3	61.5	78.5	99.3	102.4	97.2	89.4	83.9	81.4	100.5	89.8
Spain	118.2	78.3	97.1	111.3	123.8	121.1	120.9	119.1	117.5	120.4	148.1	142.7
Sweden	59.3	53.8	55.1	58.0	64.2	62.3	61.9	60.4	59.5	56.2	62.8	58.4
Switzerland	42.4	43.2	43.7	43.0	42.9	42.9	41.6	42.5	40.3	40.1	43.9	41.8
United Kingdom	118.9	103.2	107.4	103.3	113.3	112.6	119.6	119.4	116.0	118.8	151.2	142.6
United States	138.8	130.5	132.3	135.8	135.5	136.9	138.8	135.4	137.3	136.1	159.9	148.1
Average	96.1	83.4	92.7	94.3	98.9	97.4	97.5	94.6	92.6	92.7	109.4	103.4

The purpose and intent of this paper are to argue that budget deficit is not a temporary anomaly, but a very normal and enduring situation, at least in the major countries. Therefore, the pursuit of balanced budget by unnecessarily reducing government deficit and government debt in the name of sound finances is an obstacle to the stable growth of each country's economy.

5. Conclusion

In this paper I have investigate the role of budget deficit in a growing economy in which consumers hold money for the reason of liquidity using a continuous time dynamic model when consumers live forever. The main results are as follows.

- Budget deficit is necessary to achieve full employment under constant prices.
- If the actual budget deficit is greater than the value which is necessary and sufficient for full employment under constant prices, an inflation occurs.
- If the actual budget deficit is smaller than the value at which full employment is achieved under constant prices, a recession occurs.

It is not assumed that the budget deficit must later be made up by budget surplus.

In recent years, I have been interested in the issue of budget deficit and government debt, not specifically as a policy to recover from recession, but to prove that budget deficit is necessary to achieve full employment without inflation nor deflation in a steady state. The key to this is to consider a growing economy and the fact that people hold money for liquidity or other reasons. This paper develops the argument using a model in which people live forever. I have considered a continuous time dynamic model. It seems to be more general than a discrete time model. If similar conclusions can be reached in various models, a model in which people live forever may be easier to handle than a model in which generations overlap.

The budget deficit is accumulated as money or government debt, and it corresponds to assets held by the private sector. Private holdings of assets are based on the assumption that the state will last forever, at least as far as one can imagine. If the destruction of the state is foreseen, there is no need to worry, since all assets must be consumed by that date, so that demand will increase and the budget will be required to run a surplus. Unless such a national doom is foreseen, there is no need to be concerned about government debt. A high budget deficit leads to inflation; a low deficit leads to recession. That's all there is to it.

Budget deficit is not only effective in pulling the economy out of recession, they are even necessary to keep growth going without causing either recession or inflation.

For the past 10 to 20 years, Japan has been running budget deficit and yet has been unable to boost its economy and increase its growth rate despite low interest rates. This is due to the fact that despite the apparent budget deficit, the government has not necessarily spent enough, and the consumption tax hike was implemented even though the economy was still recovering. In my opinion, even a moderate budget deficit may not solve the lack of demand, since the current propensity to consume among the Japanese is very small. However, this is a subject for future research.

The diachronic budgetary constraints of government that are referred to when considering budget deficit and government debt do not exist as long as the state/government is assumed to last forever. Since neither the earth nor the sun is eternal, neither the nation nor humanity is eternal, but no nation operates with this in mind. If the end of the human race is predicted, people will spend their savings, and the propensity to consume will increase, eliminating the need for budget deficit to maintain the economy, and the government debt will disappear (along with the human race). Government debt will be eliminated by spending savings, not by taxation. There is nothing to worry about.

The accumulation of government debt (and corresponding private financial assets) does not cause immediate inflation. Inflation may occur if those assets are used to increase consumption, but then the government debt will decrease as the economy improves and budget deficit is no longer needed.

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Conflict of interest

The author claims that the manuscript is completely original. The author also declares no conflict of interest.

Appendix

A1. When money is produced by capital and labor.

Suppose that money is produced by capital and labor, and the value of money equals its production cost. The production function is

$$Y_t + \frac{\bar{M}_t}{p_t} = K_1^{\beta}\left(\theta \widetilde{K}_1 L_1\right)^{1-\beta}$$

$\bar{M}_t$ is the supply of money in Period t. It is the total cost to produce money in Period t. The market equilibrium condition for the goods under full employment is

$$G_t - \tau w_t L^f + \alpha\left[(1-\tau)w_t L^f + (r_t - g)B_t\right] + p_t g K_t = (1-\tau)w_t L^f + p_t r_t K_t - \bar{M}_t \tag{A1}$$

The condition for the money market equilibrium is that the supply of money equals an increase in money holding. Therefore,

$$\bar{M}_t = gM_t$$

By the same procedure that led to (11), (A.1) is rewritten as

$$G_t - \tau w_t L^f = 0$$

Thus, if money as well as goods are produced by capital and labor, budget deficit is not necessary for full employment under constant prices.

However, if money circulates with a value greater than the cost of production, a budget deficit will be necessary. In that case we have

$$\bar{M}_1 < gM_t$$

Then,

$$gM_t - \bar{M}_1$$

is the *seigniorage*. A moderate seigniorage to economic growth is necessary unless the production of money is quite costly.

A2. Government debt and Debt-GDP ratio.

In this appendix, I consider the case where financial assets are held not in money but in interest-producing government bonds which have almost the same liquidity as money.

Let $i<\tilde{r}$ be the interest rate of the bonds. Then, (2) is rewritten as follows;

$$b_{t+\Delta t} = (1 + r_{t+\Delta t}\Delta t)(s_{t+\Delta t} - m_{t+\Delta t}) + (1 + i\Delta t)m_{t+\Delta t}$$

The individual bond holding is denoted by $m_{t+\Delta t}$. $b_{t+\Delta t}$ is the savings. The budget constraint for consumers is

$$p_t c_t \Delta t + \frac{b_{t+\Delta t}}{1 + r_{t+\Delta t}\Delta t} + \frac{(r_{t+\Delta t} - i)\Delta t}{1 + r_{t+\Delta t}\Delta t} m_{t+\Delta t} = (1-\tau)w_t l_t \Delta t + b_t.$$

Then, we get

$$\frac{\partial b_t}{\partial t} = (1-\tau)w_t l_t - p_t c_t - (r_t - i)m_t + r_t b_t.$$

The total bond holding is

$$M_t = L^f m_t = \frac{1-\alpha}{r_t - i}[(1-\tau)w_t + (r_t - g)b_t]L^f = \frac{1-\alpha}{r_t - i}\left[(1-\tau)w_t L^f + (r_t - g)B_t\right]$$

B_t is the total savings. From this

$$(r_t - i)M_t = (1-\alpha)\left[(1-\tau)w_t L^f + (r_t - g)B_t\right]$$

And (11) is rewritten as

$$G_t - \tau w_t L^f + iM_t = gM_t$$

The left hand side of this equation is the budget deficit including interest payments on the government bonds. Thus, we obtain the same conclusion as that without interest on the bonds.

In a steady-state growth path the GDP increase at the rate of g. The government debt M_t also increases at the rate of g. Therefore, in a steady-state growth path the debt-GDP ratio is constant. Let $r_t = \tilde{r}$. In this case, (13) is rewritten as

$$M_t = (1-\alpha)\frac{1}{\tilde{r} - i}(1-\tau)w_t L^f + (1-\alpha)\frac{\tilde{r} - g}{\tilde{r} - i}B_t$$

(15) and (16) are rewritten as

$$B_t = \frac{(1-\alpha)\frac{1}{\tilde{r} - i}(1-\tau)w_t L^f + p_t\widetilde{K}_t}{1 - (1-\alpha)\frac{\tilde{r} - g}{\tilde{r} - i}}$$

and

$$M_t = \frac{(1-\alpha)\frac{1}{\tilde{r} - i}(1-\tau)w_t L^f + p_t\widetilde{K}_t}{1 - (1-\alpha)\frac{\tilde{r} - g}{\tilde{r} - i}} - p_t\widetilde{K}_t = \frac{(1-\alpha)\frac{1}{\tilde{r} - i}(1-\tau)w_t L^f + (1-\alpha)\frac{\tilde{r} - g}{\tilde{r} - i}p_t\widetilde{K}_t}{1 - (1-\alpha)\frac{\tilde{r} - g}{\tilde{r} - i}}$$

$$= (1-\alpha)\frac{(1-\tau)w_t L^f + (\tilde{r} - g)p_t\widetilde{K}_t}{\tilde{r} - i - (1-\alpha)(\tilde{r} - g)} = (1-\alpha)\frac{(1-\tau)w_t L^f + (\tilde{r} - g)p_t\widetilde{K}_t}{\alpha\tilde{r} + (1-\alpha)g - i}.$$

Since

$$\tilde{r} = g + \delta$$

the denominator of M_t is

$$\alpha\tilde{r} + (1-\alpha)g - i = g - i + \alpha\delta$$

It must be positive.

Under inflation g is replaced by $g+\pi + g\pi$, where π is the inflation rate. By economic growth M_t increases at the (nominal) growth rate. The GDP also grows at the same rate. So, the debt-GDP ratio should be constant, and as the larger the value of α (the propensity to consume) is, the smaller the debt-GDP ratio is.

A3

The Lagrange function for maximization of (1) under the budget constraint (3) is written as follows;

$$\mathcal{L}=\sum_{t=0}^{\infty}\left(\frac{1}{1+\delta\Delta t}\right)^{\frac{t}{\Delta t}}u\left(c_t,\frac{m_{t+\Delta t}}{p_t}\right)\Delta t$$

$$+\sum_{t=0}^{\infty}\lambda_t[(1+r_{t+\Delta t}\Delta t)(1-\tau)w_tl_t\Delta t-(1+r_{t+\Delta t}\Delta t)p_tc_t\Delta t-r_{t+\Delta t}\Delta tm_{t+\Delta t}+(1+r_{t+\Delta t}\Delta t)b_t-b_{t+\Delta t}]$$

λ_t for $t=0,1,\cdots,\infty$ are the Lagrange multipliers. The first order conditions for utility maximization over infinite periods are

$$\frac{\partial\mathcal{L}}{\partial c_t}=\Delta t\left[\left(\frac{1}{1+\delta\Delta t}\right)^{\frac{t}{\Delta t}}\frac{\alpha}{c_t}-\lambda_t(1+r_{t+\Delta t}\Delta t)p_t\right]=0,t=0,1,\cdots,\infty$$

and

$$\frac{\partial\mathcal{L}}{\partial\frac{m_{t+\Delta t}}{p_t}}=\Delta t\left[\left(\frac{1}{1+\delta\Delta t}\right)^{\frac{t}{\Delta t}}\frac{(1-\alpha)p_t}{m_{t+\Delta t}}-\lambda_tr_{t+\Delta t}p_t\right]=0,t=0,1,\cdots,\infty$$

They mean

$$\left(\frac{1}{1+\delta\Delta t}\right)^{\frac{t}{\Delta t}}\frac{\alpha}{c_t}-\lambda_t(1+r_{t+\Delta t}\Delta t)p_t=0$$

and

$$\left(\frac{1}{1+\delta\Delta t}\right)^{\frac{t}{\Delta t}}\frac{(1-\alpha)p_t}{m_{t+\Delta t}}-\lambda_tr_{t+\Delta t}p_t=0$$

Suppose

$$\Delta t\to 0$$

then, the first order conditions are written as follows;

$$e^{-\delta t}\frac{\alpha}{c_t}-\lambda_tp_t=0$$

and

$$e^{-\delta t}(1-\alpha)\frac{p_t}{m_t}-\lambda_tr_tp_t=0$$

From them,

$$p_t c_t = e^{-\delta t} \frac{\alpha}{\lambda_t} \tag{A2}$$

and

$$m_t = e^{-\delta t} \frac{1-\alpha}{r_t \lambda_t} \tag{A3}$$

are derived. By the Lagrange function,

$$\frac{\partial \mathcal{L}}{\partial b_t} = \lambda_t(1 + r_{t+\Delta t}\Delta t) - \lambda_{t-\Delta t} = 0$$

From this,

$$\lambda_t - \lambda_{t-\Delta t} = -r_{t+\Delta t}\lambda_t \Delta t$$

Therefore,

$$\frac{\lambda_t - \lambda_{t-\Delta t}}{\Delta t} = -r_{t+\Delta t}\lambda_t$$

Suppose

$$\Delta t \to 0$$

Then,

$$\frac{\partial \lambda_t}{\partial t} = -r_t \lambda_t \tag{A4}$$

By (5), (A.2) and (A.3), we get

$$c_t = \frac{\alpha}{p_t}[(1-\tau)w_t l_t + (r_t - g)b_t] \tag{6}$$

and

$$m_t = \frac{1-\alpha}{r_t}[(1-\tau)w_t l_t + (r_t - g)b_t] \tag{7}$$

They mean

$$\frac{e^{-\delta t}}{\lambda_t} = (1-\tau)w_t l_t + (r_t - g)b_t$$

Thus,

$$\lambda_t = \frac{\alpha}{p_t c_t} e^{-\delta t} = \frac{1-\alpha}{r_t m_t} e^{-\delta t} \tag{A5}$$

Assume that r_t and p_t are constant, and c_t and m_t are increasing at the rate of g. Differentiating (A.4) with respect to t,

$$r_t m_t \frac{\partial \lambda_t}{\partial t} + r_t \lambda_t \frac{\partial m_t}{\partial t} = r_t m_t \frac{\partial \lambda_t}{\partial t} + r_t \lambda_t g m_t = -\delta(1-\alpha)e^{-\delta t} = -\delta r_t m_t \lambda_t$$

This means

$$\frac{\partial \lambda_t}{\partial t} + \lambda_t g = -\delta \lambda_t$$

Therefore,

$$\frac{\partial \lambda_t}{\partial t} = -(g+\delta)\lambda_t$$

Then, from (A.4) we find

$$r_t = g + \delta \tag{8}$$

This is the equilibrium interest rate (or the rate of return of the capital) in our model.

Analysis by Hamiltonian methods

Let us consider the Hamiltonian method. In our model with (4) the present value Hamiltonian is written as follows;

$$H_t = e^{-\delta t} u\left(c_t, \frac{m_t}{p_t}\right) + \lambda_t[(1-\tau)w_t l_t - p_t c_t - r_t m_t + r_t b_t]$$

The first order conditions are

$$\frac{\partial H_t}{\partial c_t} = e^{-\delta t}\frac{\alpha}{c_t} - \lambda_t p_t = 0$$

and

$$\frac{\partial H_t}{\partial \frac{m_t}{p_t}} = e^{-\delta t}\frac{1-\alpha}{m_t} p_t - \lambda_t r_t p_t = 0$$

These are equivalent to (A.2) and (A.3). The costate equation is

$$\frac{\partial H_t}{\partial b_t} = r_t \lambda_t = -\frac{\partial \lambda_t}{\partial t}$$

This is equivalent to (A.4).

On the other hand, the current value Hamiltonian is defined by

$$H_t^c = u\left(c_t, \frac{m_t}{p_t}\right) + \mu_t[(1-\tau)w_t l_t - p_t c_t - r_t m_t + r_t b_t]$$

where

$$\mu_t = e^{\delta t}\lambda_t$$

Then,

$$\frac{\partial \mu_t}{\partial t} = \delta e^{\delta t}\lambda_t + e^{\delta t}\frac{\partial \lambda_t}{\partial t} = \delta \mu_t + e^{\delta t}\frac{\partial \lambda_t}{\partial t}$$

or

$$\delta \mu_t - \frac{\partial \mu_t}{\partial t} = -e^{\delta t}\frac{\partial \lambda_t}{\partial t}$$

The first order conditions are

$$\frac{\partial H_t^c}{\partial c_t} = \frac{\alpha}{c_t} - \mu_t p_t = \frac{\alpha}{c_t} - e^{\delta t}\lambda_t p_t = 0$$

and

$$\frac{\partial H_t^c}{\partial \frac{m_t}{p_t}} = \frac{1-\alpha}{m_t}p_t - e^{\delta t}\lambda_t r_t p_t = 0$$

They are equivalent to (A.2) and (A.3). The costate equation is

$$\frac{\partial H_t^c}{\partial b_t} = r_t \mu_t = r_t e^{\delta t}\lambda_t = -e^{\delta t}\frac{\partial \lambda_t}{\partial t}$$

This means

$$\frac{\partial H_t^c}{\partial b_t} = \delta \mu_t - \frac{\partial \mu_t}{\partial t}$$

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